

The Cohomology Algebra of Polyhedral Products

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Abstract

In this paper, we compute the homology coalgebra and cohomology algebra over a field of all polyhedral products and give a duality theorem on complementary polyhedral products.

The moment-angle complexes have been studied by topologists for many years (see [19] [15]). In 1990's Davis and Januszkiewicz [8] introduced toric manifolds which are studied intensively by algebraic geometers. They constructed a quasi-toric manifold from a moment-angle complex the topology of which is complicated and getting more attentions by topologists lately (see [11] [14] [4] [18] [10]). Recently a lot of work has been done on generalizing the moment-angle complex $\mathcal{Z}(K; D^2, S^1)$ to space pairs (X, A) (see [2], [3], [12], [16]). Such a space $\mathcal{Z}(K; X, A)$ is called a polyhedral product. But in general, the (co)homology of a polyhedral product is not known.

In Theorem 1.6, we construct a chain complex that is chain homotopic equivalent to the singular chain complex of a polyhedral product and compute the (co)homology group. In Theorem 2.5, we give a duality theorem on the complementary polyhedral product. In Theorem 3.4, we compute the coproduct of the homology coalgebra of a polyhedral product by chain complex approximations. In Theorem 4.3, we restate Theorem 3.4 from the point of view of diagonal tensor product. In Theorem 4.4 to Theorem 4.7, we compute the cohomology algebra of some typical polyhedral products.

The paper ends in Example 4.8 which solve an unknown problem suggested by Bahri.

1 The (co)homology group of a polyhedral product

Notations and Conventions In this paper, \mathbb{F} is a field. All objects (groups, (co)chain complexes, (co)algebras, etc.) are graded vector spaces over \mathbb{F} . Simplicial and singular (co)homology means (co)homology over \mathbb{F} and \otimes means $\otimes_{\mathbb{F}}$. So base and dual objects always exist and $H_*(C_1 \otimes C_2, d_1 \otimes d_2) = H_*(C_1, d_1) \otimes H_*(C_2, d_2)$. Since cohomology is always the dual vector space of homology, all the proofs of the cohomology case are omitted in this paper.

For a positive integer m , $[m]$ denotes the set $\{1, 2, \dots, m\}$.

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A simplicial complex K with vertex set S is a set of subsets of S such that if $\tau \in K$ and $\sigma \subset \tau$, then $\sigma \in K$ and $\{s\} \in K$ for every $s \in S$. 2^S is the full simplicial complex consisting of all subsets of S . All simplicial complexes have the empty simplex ϕ as the unique -1 -dimensional simplex except the void complex $\{\}$ that has no simplex.

Definition 1.1 Let $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ be a sequence of topological space pairs and K be a simplicial complex with vertex set a subset of $[m]$. The polyhedral product $\mathcal{Z}(K; X, A)$ is the topological space defined as follows. For a subset σ of $[m]$, define

$$D(\sigma) = Y_1 \times \cdots \times Y_n, \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Then $\mathcal{Z}(K; X, A) = \cup_{\sigma \in K} D(\sigma)$.

Notice that different from the usual definition, in a polyhedral product $\mathcal{Z}(K; X, A)$, the vertex set of K may be a proper subset of $[m]$. This will give a complete conclusion for duality.

Definition 1.2 A system V with index set Λ is a direct sum group $V = \oplus_{\alpha \in \Lambda} V_\alpha$ such that every V_α is a graded group. For a subset Γ of Λ , the system $U = \oplus_{\gamma \in \Gamma} V_\gamma$ is called the subsystem of V with index set Γ . For two systems $V = \oplus_{\alpha \in \Lambda} V_\alpha$ with index set Λ and $W = \oplus_{\beta \in \Pi} W_\beta$ with index set Π , their direct sum system $V \oplus W = (\oplus_{\alpha \in \Lambda} V_\alpha) \oplus (\oplus_{\beta \in \Pi} W_\beta)$ is the system with index set $\Lambda \sqcup \Pi$ (\sqcup disjoint union). It is obvious that both V and W are subsystems of $V \oplus W$. For two systems $V = \oplus_{\alpha \in \Lambda} V_\alpha$ and $W = \oplus_{\alpha \in \Lambda} W_\alpha$ with the same index set Λ , their diagonal tensor product $V \hat{\otimes} W$ is the system $V \hat{\otimes} W = \oplus_{\alpha \in \Lambda} V_\alpha \otimes W_\alpha$ with index set Λ .

Definition 1.3 Let K be a simplicial complex with vertex set a subset of $[m]$ and Σ, Ω be two subsets of $[m]$. Define $I(K; \Sigma, \Omega) = \{(\sigma, \omega) \mid \sigma \in K, \sigma \subset \Sigma, \omega \subset \Omega, \sigma \cap \omega = \phi\}$ and $\bar{I}(K; \Sigma, \Omega) = \{(\sigma, \omega) \in I(K; \Sigma, \Omega) \mid \omega \neq \phi\}$. For $(\sigma, \omega) \in I(K; \Sigma, \Omega)$, $K_{\sigma, \omega} = \{\eta \in \text{link}_K \sigma \mid \eta \subset \omega\}$, where $\text{link}_K \sigma = \{\tau \mid \sigma \cup \tau \in K, \sigma \cap \tau = \phi\}$ is the link of σ .

The simplicial homology system $H_*(K; \Sigma, \Omega)$ of K with index set $I(K; \Sigma, \Omega)$ is defined as follows.

$$H_*(K; \Sigma, \Omega) = \oplus_{(\sigma, \omega) \in I(K; \Sigma, \Omega)} H_{\sigma, \omega}^{\sigma, \omega}(K), \quad H_{\sigma, \omega}^{\sigma, \omega}(K) = \tilde{H}_{*-1}(K_{\sigma, \omega}),$$

where \tilde{H}_{*-1} is the reduced simplicial homology over \mathbb{F} . $\bar{H}_*(K; \Sigma, \Omega)$ is the subsystem of $H_*(K; \Sigma, \Omega)$ with index set $\bar{I}(K; \Sigma, \Omega)$.

Dually, the simplicial cohomology system $H^*(K; \Sigma, \Omega)$ of K with index set $I(K; \Sigma, \Omega)$ is defined as follows.

$$H^*(K; \Sigma, \Omega) = \oplus_{(\sigma, \omega) \in I(K; \Sigma, \Omega)} H_{\sigma, \omega}^{\sigma, \omega}(K), \quad H_{\sigma, \omega}^{\sigma, \omega}(K) = \tilde{H}^{*-1}(K_{\sigma, \omega}),$$

where \tilde{H}^{k-1} is the reduced simplicial cohomology over \mathbb{F} . $\bar{H}^*(K; \Sigma, \Omega)$ is the subsystem of $H^*(K; \Sigma, \Omega)$ with index set $\bar{I}(K; \Sigma, \Omega)$.

It is obvious that there are system direct sum decompositions

$$H_*(K; \Sigma, \Omega) = H_*(K; \Sigma, \phi) \oplus \bar{H}_*(K; \Sigma, \Omega), \quad H^*(K; \Sigma, \Omega) = H^*(K; \Sigma, \phi) \oplus \bar{H}^*(K; \Sigma, \Omega).$$

Definition 1.4 Let $i: A \rightarrow B$ be a graded group homomorphism. Since A is a vector space over \mathbb{F} , we can take

a subgroup \mathfrak{i} of A such that $A = \ker i \oplus \mathfrak{i}$ (so $\mathfrak{i} \cong \text{coim } i$). Define quotient graded group (depends on the choice of \mathfrak{i}) $A \cup_i B = (A \oplus B)/C$ with $C = \{(a, -i(a)) \mid a \in \mathfrak{i}\}$. It is obvious that the isomorphism class of $A \cup_i B$ is independent of the choice of \mathfrak{i} and both A and B are subgroup of $A \cup_i B$.

Definition 1.5 Let $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ be a sequence of topological space pairs and $i_k: H_*(A_k) \rightarrow H_*(X_k)$ be the singular homology homomorphism induced by the inclusion map and $i_k^*: H^*(X_k) \rightarrow H^*(A_k)$ be the dual of i_k . Suppose a fixed direct sum decomposition $H_*(A_k) = \ker i_k \oplus \text{coim } i_k$, $H_*(X_k) = \text{im } i_k \oplus \text{coker } i_k$ and the dual decomposition $H^*(X_k) = \ker i_k^* \oplus \text{coim } i_k^*$, $H^*(A_k) = \text{im } i_k^* \oplus \text{coker } i_k^*$ are given for $k = 1, \dots, m$. Then we have dual groups

$$\begin{aligned} T_*(X, A) &= (H_*(A_1) \cup_{i_1} H_*(X_1)) \otimes \cdots \otimes (H_*(A_m) \cup_{i_m} H_*(X_m)), \\ T^*(X, A) &= (H^*(X_1) \cup_{i_1^*} H^*(A_1)) \otimes \cdots \otimes (H^*(X_m) \cup_{i_m^*} H^*(A_m)). \end{aligned}$$

For any subsets σ, ω of $[m]$ such that $\sigma \cap \omega = \emptyset$, define subgroup $T_*^{\sigma, \omega}(X, A)$ of $T_*(X, A)$ and subgroup $T_{\sigma, \omega}^*(X, A)$ of $T^*(X, A)$ as follows

$$\begin{aligned} T_*^{\sigma, \omega}(X, A) &= T_1 \otimes \cdots \otimes T_m, \quad T_k = \begin{cases} \text{coker } i_k & \text{if } k \in \sigma, \\ \ker i_k & \text{if } k \in \omega, \\ \text{im } i_k = \text{coim } i_k & \text{otherwise,} \end{cases} \\ T_{\sigma, \omega}^*(X, A) &= T^1 \otimes \cdots \otimes T^m, \quad T^k = \begin{cases} \ker i_k^* & \text{if } k \in \sigma, \\ \text{coker } i_k^* & \text{if } k \in \omega, \\ \text{im } i_k^* = \text{coim } i_k^* & \text{otherwise.} \end{cases} \end{aligned}$$

Since the (co)homology is over the field \mathbb{F} , we have

$$\Sigma = \{k \mid \text{coker } i_k \neq 0\} = \{k \mid \ker i_k^* \neq 0\}, \quad \Omega = \{k \mid \ker i_k \neq 0\} = \{k \mid \text{coker } i_k^* \neq 0\}.$$

It is obvious that $T_*^{\sigma, \omega}(X, A) = 0$ and $T_{\sigma, \omega}^*(X, A) = 0$ if $\sigma \not\subset \Sigma$ or $\omega \not\subset \Omega$. Let $I(X, A) = \{(\sigma, \omega) \mid \sigma \subset \Sigma, \omega \subset \Omega, \sigma \cap \omega = \emptyset\}$.

Then $T_*(X, A) = \bigoplus_{(\sigma, \omega) \in I(X, A)} T_*^{\sigma, \omega}(X, A)$ and $T^*(X, A) = \bigoplus_{(\sigma, \omega) \in I(X, A)} T_{\sigma, \omega}^*(X, A)$ are systems with index set $I(X, A)$.

For a simplicial complex K with vertex set a subset of $[m]$, the homology generating system $T_*(K; X, A)$ of K with respect to (X, A) is the subsystem of $T_*(X, A)$ with index set $I(K; X, A) = I(K; \Sigma, \Omega)$, where Σ and Ω are as above and $I(K; \Sigma, \Omega)$ is as in Definition 1.3. $\bar{T}_*(K; X, A)$ is the subsystem of $T_*(K; X, A)$ with index set $\bar{I}(K; X, A) = \bar{I}(K; \Sigma, \Omega)$ and $\hat{T}_*(K; X, A)$ is the subsystem of $T_*(K; X, A)$ with index set $I(K; \Sigma, \emptyset)$. It is obvious that $T_*(K; X, A) = \hat{T}_*(K; X, A) \oplus \bar{T}_*(K; X, A)$.

Dually, the cohomology generating system $T^*(K; X, A)$ of K with respect to (X, A) is the subsystem of $T^*(X, A)$ with index set $I(K; X, A)$. $\bar{T}^*(K; X, A)$ is the subsystem of $T^*(K; X, A)$ with index set $\bar{I}(K; X, A)$ and $\hat{T}^*(K; X, A)$ is the subsystem of $T^*(K; X, A)$ with index set $I(K; \Sigma, \emptyset)$. It is obvious that $T^*(K; X, A) = \hat{T}^*(K; X, A) \oplus \bar{T}^*(K; X, A)$.

By definition, $T_*(X, A) = T_*(2^{[m]}; X, A)$ and $T^*(X, A) = T^*(2^{[m]}; X, A)$.

Theorem 1.6 Let $M = \mathcal{Z}(K; X, A)$, $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ be a polyhedral product such that every A_k is either an open subspace of X_k or a deformation retract of an open subspace of X_k and everything else be as in Definition 1.3 and Definition 1.5. Then the (co)homology group (over \mathbb{F}) of the polyhedral product M is isomorphic to the diagonal tensor product of the simplicial (co)homology system of K with index set $I(K; X, A)$ and the (co)homology generating

system of K with respect to (X, A) , i.e., $H_*(M) \cong H_*(K; \Sigma, \Omega) \widehat{\otimes} T_*(K; X, A)$ and $H^*(M) \cong H^*(K; \Sigma, \Omega) \widehat{\otimes} T^*(K; X, A)$. Precisely,

$$H_*(M) \cong \oplus_{(\sigma, \omega) \in I(K; X, A)} H_*^{\sigma, \omega}(M), \quad H_*^{\sigma, \omega}(M) = H_*^{\sigma, \omega}(K) \otimes T_*^{\sigma, \omega}(X, A),$$

$$H^*(M) \cong \oplus_{(\sigma, \omega) \in I(K; X, A)} H^*_{\sigma, \omega}(M), \quad H^*_{\sigma, \omega}(M) = H^*_{\sigma, \omega}(K) \otimes T^*_{\sigma, \omega}(X, A),$$

where $H_*^{\sigma, \omega}(K)$, $H^*_{\sigma, \omega}(K)$, $T_*^{\sigma, \omega}(X, A)$, $T^*_{\sigma, \omega}(X, A)$ are as in Definition 1.3 and Definition 1.5.

The diagonal tensor product is natural in the following sense. If K is a simplicial subcomplex of L , then $I(K; X, A)$ is a subset of $I(L; X, A)$ and so $T_*(K; X, A)$ is a subsystem of $T_*(L; X, A)$. For $(\sigma, \omega) \in I(K; X, A)$, let $i_{\sigma, \omega}: H_*^{\sigma, \omega}(K) \rightarrow H_*^{\sigma, \omega}(L)$, $i: H_*(K) \rightarrow H_*(L)$ be induced by the inclusion map, then we have the following commutative diagram

$$\begin{array}{ccc} \oplus H_*^{\sigma, \omega}(K) \otimes T_*^{\sigma, \omega}(X, A) & \xrightarrow{\oplus i_{\sigma, \omega} \otimes 1} & \oplus H_*^{\sigma, \omega}(L) \otimes T_*^{\sigma, \omega}(X, A) \\ \parallel \wr & & \parallel \wr \\ H_*(\mathcal{Z}(K; X, A)) & \xrightarrow{i} & H_*(\mathcal{Z}(L; X, A)), \end{array}$$

where 1 denotes the identity. Dually, we have the following commutative diagram

$$\begin{array}{ccc} \oplus H^*_{\sigma, \omega}(L) \otimes T^*_{\sigma, \omega}(X, A) & \xrightarrow{\oplus i^*_{\sigma, \omega} \otimes 1} & \oplus H^*_{\sigma, \omega}(K) \otimes T^*_{\sigma, \omega}(X, A) \\ \parallel \wr & & \parallel \wr \\ H^*(\mathcal{Z}(L; X, A)) & \xrightarrow{i^*} & H^*(\mathcal{Z}(K; X, A)), \end{array}$$

where $i^*_{\sigma, \omega} = 0$ if $(\sigma, \omega) \in I(L; X, A)$, $(\sigma, \omega) \notin I(K; X, A)$.

Proof We may suppose that every A_k is an open subspace of X_k , otherwise, replace A_k by the open subspace of X_k of which A_k is a deformation retract.

For a topological space Y , denote by $(S_*(Y), d)$ the singular chain complex over \mathbb{F} of Y . Take a representative in $S_*(A_k)$ for all homology classes in a base of $\ker i_k$ and $\text{coim } i_k$ and denote the two sets of such representatives respectively by \mathfrak{k}_k and \mathfrak{i}_k . Take a representative in $S_*(X_k)$ for all homology classes in a base of $\text{coker } i_k$ and denote the set of these representatives by \mathfrak{c}_k . For every $x \in \mathfrak{k}_k$, take a fixed $y \in S_*(X_k)$ such that $dy = x$ and denote by \mathfrak{q}_k the set of all such y . Let $U_*(k) = \mathbb{F}(\mathfrak{k}_k \cup \mathfrak{i}_k \cup \mathfrak{c}_k \cup \mathfrak{q}_k)$ and $V_*(k) = \mathbb{F}(\mathfrak{k}_k \cup \mathfrak{i}_k)$, where $\mathbb{F}(\mathfrak{s})$ is the vector space over \mathbb{F} with base \mathfrak{s} . Then $(U_*(k), d)$ and $(V_*(k), d)$ are respectively chain subcomplexes of $(S_*(X_k), d)$ and $(S_*(A_k), d)$ that make the following diagram commutative

$$\begin{array}{ccc} (V_*(k), d) & \xrightarrow{i_k} & (U_*(k), d) \\ i_k \downarrow & & i_k \downarrow \\ (S_*(A_k), d) & \xrightarrow{i_k} & (S_*(X_k), d), \end{array}$$

where we use i_k to denote all inclusion of chain complexes. The two vertical inclusions are chain homotopy equivalences that induce homology group isomorphisms.

Let $(C_*(m), d) = (U_*(1) \otimes \cdots \otimes U_*(m), d \otimes \cdots \otimes d)$. Construct chain subcomplex $(C_*(M), d)$ of $(C_*(m), d)$ as follows. For a subset $\sigma \subset [m]$, define $(C_*(\sigma), d) = (W_*(1) \otimes \cdots \otimes W_*(m), d \otimes \cdots \otimes d)$, where $W_*(k) = U_*(k)$ if $k \in \sigma$ and $W_*(k) = V_*(k)$ if $k \notin \sigma$. Then for $\sigma \subset \sigma'$, $(C_*(\sigma), d)$ is a chain subcomplex of $(C_*(\sigma'), d)$ and for all $\sigma, \tau \subset [m]$, $C_*(\sigma) \cap C_*(\tau) =$

$C_*(\sigma \cap \tau)$. Define $C_*(M) = \bigoplus_{\sigma \in K} C_*(\sigma)$ (not $\oplus!$). Then $(C_*(M), d)$ is a chain subcomplex of $(S_*(M), d)$. Denote the inclusion by $i_M: (C_*(M), d) \rightarrow (S_*(M), d)$.

Now we prove $i_M: (C_*(M), d) \rightarrow (S_*(M), d)$ is a chain homotopy equivalence, i.e., the homology homomorphism $(i_M)_*: H_*(C_*(M), d) \rightarrow H_*(M)$ is an isomorphism. We use induction on the number of maximal simplices (a maximal simplex is not the proper face of any simplex) of K . If K has only one maximal simplex σ , i.e., $K = 2^\sigma$ for a subset σ of $[m]$, then i_M is the composite of the following inclusion chain homotopy equivalences

$$i_M: (C_*(M), d) = (C_*(\sigma), d) \rightarrow (S_*(Y_1) \otimes \cdots \otimes S_*(Y_m), d \otimes \cdots \otimes d) \rightarrow (S_*(Y_1 \times \cdots \times Y_m), d) = (S_*(D(\sigma)), d),$$

where $D(\sigma)$ is as defined in Definition 1.1, the first inclusion is the tensor product $i_1 \otimes \cdots \otimes i_m$ and the second is the Eilenberg-Zilber chain homotopy equivalence. Suppose i_N is a chain homotopy equivalence for all polyhedral product $N = \mathcal{Z}(L; X, A)$ such that L has $\leq u$ maximal simplices and K is a simplicial complex with $u+1$ maximal simplices $\sigma_1, \dots, \sigma_{u+1}$. Denote by K_1 the simplicial complex with maximal simplices $\sigma_1, \dots, \sigma_u$, $K_2 = 2^{\sigma_{u+1}}$, $K_3 = K_1 \cap K_2$ and $M_i = \mathcal{Z}(K_i; X, A)$. Then by induction hypothesis, i_{M_k} is a chain homotopy equivalence for $k = 1, 2, 3$. From the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow & C_*(M_3) & \rightarrow & C_*(M_1) \oplus C_*(M_2) & \rightarrow & C_*(M) & \rightarrow 0 \\ & i_{M_3} \downarrow & & i_{M_1} \oplus i_{M_2} \downarrow & & i'_M \downarrow & \\ 0 \rightarrow & S_*(M_3) & \rightarrow & S_*(M_1) \oplus S_*(M_2) & \rightarrow & (S_*(M_1) \oplus S_*(M_2))/S_*(M_3) & \rightarrow 0 \end{array}$$

we get a chain homotopy equivalence i'_M . By excision axiom, the inclusion $i_S: (S_*(M_1) \oplus S_*(M_2))/S_*(M_3) \rightarrow S_*(M)$ is a chain homotopy equivalence and so $i_M = i_S i'_M$ is a chain homotopy equivalence. The induction is completed. Thus, $H_*(C_*(M), d) \cong H_*(M)$ for all M .

$C_*(M)$ has a base $\Phi = \{a_1 \otimes \cdots \otimes a_m \in C_*(M) \mid a_k \in \mathfrak{k}_k \text{ or } \mathfrak{i}_k \text{ or } \mathfrak{c}_k \text{ or } \mathfrak{q}_k\}$. Then $T_*(K; X, A)$ is a subgroup of $C_*(M)$ with base $\Lambda = \{a_1 \otimes \cdots \otimes a_m \in T_*(K; X, A) \mid a_k \in \mathfrak{k}_k \text{ or } \mathfrak{i}_k \text{ or } \mathfrak{c}_k\}$. Define equivalence relation \sim on Φ as follows. $a_1 \otimes \cdots \otimes a_m \sim b_1 \otimes \cdots \otimes b_m$ if for every $k = 1, \dots, m$, either $a_k = b_k$, or $da_k = b_k$, or $db_k = a_k$. Then every equivalence class has one and only one element in Λ , i.e., the equivalence class set Φ/\sim is in 1-1 correspondence with Λ . For every $x \in \Lambda$, let $C_*(x)$ be the subgroup of $C_*(M)$ with base $\{y \in \Phi \mid x \sim y\}$ (just the equivalence class of x). It is obvious that $(C_*(x), d)$ is a chain subcomplex of $(C_*(M), d)$ and we have a direct sum decomposition $(C_*(M), d) = \bigoplus_{x \in \Lambda} (C_*(x), d)$.

Now consider the chain complex $(C_*(x), d)$. Suppose $x = a_1 \otimes \cdots \otimes a_m$, $\sigma = \{k \mid a_k \in \mathfrak{c}_k\}$, $\omega = \{k \mid a_k \in \mathfrak{k}_k\}$. For any base element $y = b_1 \otimes \cdots \otimes b_m \in C_*(x)$, define $\eta_y = \{k \mid a_k \neq b_k\}$. By definition, $\eta_y \subset \omega$. Since $y \in C_*(M) = \bigoplus_{\tau \in K} C_*(\tau)$, there is a $\tau \in K$ such that $y \in C_*(\tau)$. This implies that $\eta_y \cup \sigma \subset \tau$. So $\eta_y \in K_{\sigma, \omega} = \{\eta \subset \omega \mid \eta \cup \sigma \in K, \eta \cap \sigma = \emptyset\}$. Conversely, for $\eta \in K_{\sigma, \omega}$, define $y_\eta = b_1 \otimes \cdots \otimes b_m$ be such that $b_k = a_k$ if $k \notin \eta$ and $db_k = a_k$ if $k \in \eta$. Then $y_\eta \sim x$ and $y_\eta \in C_*(M)$, since $\eta \cup \sigma \in K$. So the correspondence $y \rightarrow \eta_y$ is a 1-1 correspondence from the base of $C_*(x)$ to $K_{\sigma, \omega}$ and induces a graded group isomorphism $\psi: C_*(x) \rightarrow \Sigma^{|x|+1} \tilde{C}_*(K_{\sigma, \omega})$, where Σ^k means uplift the degree of a graded group by k and $\tilde{C}_*(K_{\sigma, \omega})$ is the augmented simplicial chain group with base $K_{\sigma, \omega}$. Let d_x (depends on the multi-degree $(|a_1|, \dots, |a_m|)$) be the differential on $\tilde{C}_*(K_{\sigma, \omega})$ defined by

$$d_x(\{i_1, \dots, i_s\}) = \Sigma_{k=1}^s (-1)^{|a_1|+|a_2|+\dots+|a_{i_k-1}|+k-1} \{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_s\}.$$

It is easy to check that $d_x\psi = \psi d_x$. So ψ is a chain complex isomorphism from $(C_*(x), d)$ to $(\Sigma^{|x|+1}\tilde{C}_*(K_{\sigma,\omega}), d_x)$. Let d be the usual differential of $\tilde{C}_*(K_{\sigma,\omega})$ define by $d\{i_1, \dots, i_s\} = \Sigma_{k=1}^s (-1)^{k-1} \{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_s\}$. Then the correspondence $\{i_1, \dots, i_s\} \rightarrow (-1)^\tau \{i_1, \dots, i_s\}$ with $\tau = (|a_1|+\dots+|a_{i_1-1}|)+(|a_1|+\dots+|a_{i_2-1}|)+\dots+(|a_1|+\dots+|a_{i_s-1}|)$ induces a chain complex isomorphism from $(\tilde{C}_*(K_{\sigma,\omega}), d)$ to $(\tilde{C}_*(K_{\sigma,\omega}), d_x)$. So $H_*(\tilde{C}_*(K_{\sigma,\omega}), d_x)$ is independent of d_x and $H_*(C_*(x), d) = \Sigma^{|x|+1}H_*(\tilde{C}_*(K_{\sigma,\omega}), d) = \Sigma^{|x|+1}\tilde{H}_*(K_{\sigma,\omega}) = \Sigma^{|x|}H_*^{\sigma,\omega}(K) = H_*^{\sigma,\omega}(K) \otimes \mathbb{F}(x)$, where $\mathbb{F}(x)$ is the 1-dimensional graded group generated by x with degree $|x|$.

$$\text{Thus, } H_*(M) = \oplus_{(\sigma,\omega) \in I(K;X,A)} \left(\oplus_{x \in \Lambda, x \in T_*^{\sigma,\omega}(X,A)} H_*^{\sigma,\omega}(K) \otimes \mathbb{F}(x) \right) = \oplus_{(\sigma,\omega) \in I(K;X,A)} H_*^{\sigma,\omega}(K) \otimes T_*^{\sigma,\omega}(X,A).$$

If $N = \mathcal{Z}(L; X, A)$ and K is a simplicial subcomplex of L , then $(C_*(M), d)$ is a chain subcomplex of $(C_*(N), d)$ such that $(\hat{C}_*(K_{\sigma,\omega}), d) \otimes \mathbb{F}(x)$ is a chain subcomplex of $(\hat{C}_*(L_{\sigma,\omega}), d) \otimes \mathbb{F}(x)$ for every $x \in T_*(K; X, A)$. So the diagonal tensor product is natural.

Theorem 1.7 *Let everything be as in Theorem 1.6. Let $i_M: H_*(M) \rightarrow H_*(\tilde{X})$ and $i_M^*: H^*(\tilde{X}) \rightarrow H^*(M)$ be singular homology and cohomology homomorphism induced by the inclusion map from M to \tilde{X} with $\tilde{X} = X_1 \times \dots \times X_m$. Define $\hat{H}_*(M) = \text{coim } i_M$, $\overline{H}_*(M) = \ker i_M$, $\hat{H}^*(M) = \text{im } i_M^*$ and $\overline{H}^*(M) = \text{coker } i_M^*$. Then*

$$\begin{aligned} \overline{H}_*(M) &\cong \overline{H}_*(K; \Sigma, \Omega) \hat{\otimes} \overline{T}_*(K; X, A), \quad \hat{H}_*(M) \cong \hat{T}_*(K; X, A), \\ \overline{H}^*(M) &\cong \overline{H}^*(K; \Sigma, \Omega) \hat{\otimes} \overline{T}^*(K; X, A), \quad \hat{H}^*(M) \cong \hat{T}^*(K; X, A). \end{aligned}$$

Proof Regard \tilde{X} as the polyhedral product $\tilde{X} = \mathcal{Z}(2^{[m]}; X, A)$. If $\omega \neq \phi$, then $(2^{[m]})_{\sigma,\omega} = 2^\omega$ and $H_*^{\sigma,\omega}(2^{[m]}) = 0$. If $\omega = \phi$, then $K_{\sigma,\phi} = (2^{[m]})_{\sigma,\phi} = \{\phi\}$ and $H_*^{\sigma,\phi}(K) = H_*^{\sigma,\phi}(\tilde{2}^{[m]}) = \mathbb{F}$. Identify $H_*^{\sigma,\phi}(K) \otimes T_*^{\sigma,\phi}(X, A)$ with $T_*^{\sigma,\phi}(X, A)$. Then we have $H_*(M) = (\oplus_{\sigma \in K} T_*^{\sigma,\phi}(X, A)) \oplus (\oplus_{(\sigma,\omega) \in \tilde{I}(K;X,A)} H_*^{\sigma,\omega}(K; \Sigma, \Omega) \otimes T_*^{\sigma,\omega}(X, A))$, $H_*(\tilde{X}) = \oplus_{\sigma \in 2^{[m]}} T_*^{\sigma,\phi}(X, A)$. The homomorphism induced by inclusion map $i_{\sigma,\omega}: H_*^{\sigma,\omega}(K) \rightarrow H_*^{\sigma,\omega}(2^{[m]})$ satisfies $i_{\sigma,\omega} = 0$ if $\omega \neq \phi$ and $i_{\sigma,\phi} = 1$ (1 identity) for $\sigma \in K$. So by Theorem 1.6, $\ker i_M = \overline{H}_*(K; \Sigma, \Omega) \hat{\otimes} \overline{T}_*(K; X, A)$, $\text{coim } i_M \cong \hat{T}_*(K; X, A)$.

Example 1.8 For $q \leq r$, the standard inclusion of the sphere $S^q \hookrightarrow \mathbb{R}^{r+1}$ induces an inclusion map $\theta: S^q \rightarrow S^{r+1}$, where we regard S^{r+1} as one-point compactification of \mathbb{R}^{r+1} . So (S^{r+1}, S^q) is a space pair. For $q_k \leq r_k$, $k = 1, \dots, m$, let $M = \mathcal{Z}_K \left(\begin{smallmatrix} r_1+1 & \dots & r_m+1 \\ q_1 & \dots & q_m \end{smallmatrix} \right) = \mathcal{Z}(K; X, A)$ be the polyhedral product with every space pair $(X_k, A_k) = (S^{r_k+1}, S^{q_k})$. Then $I(K; X, A) = I(K; [m], [m]) = \{(\sigma, \omega) \mid \sigma \in K, \sigma \cap \omega = \phi\}$.

We first compute the (co)homology generating system of (X, A) . Since all the graded groups $\ker i_k$, $\text{coker } i_k$, $\text{coim } i_k = \text{im } i_k$ and their dual groups are one dimensional, we use the degree to represent the unique generator of the group. Then both $H_*(A_k) \cup_{i_k} H_*(X_k)$ and $H^*(X_k) \cup_{i_k^*} H^*(A_k)$ are graded groups generated by three generators $0, q_k, r_k+1$. For every $(\sigma, \omega) \in I(K; [m], [m])$, $T_*^{\sigma,\omega}(X, A)$ and $T_*^{\sigma,\omega}(X, A)$ are 1-dimensional generated by $T(\sigma, \omega) = n_1 \otimes \dots \otimes n_m$ such that $n_k = r_k+1$ if $k \in \sigma$; $n_k = q_k$ if $k \in \omega$; $n_k = 0$ otherwise. So $|T(\sigma, \omega)| = \Sigma_{i \in \sigma} (r_i + 1) + \Sigma_{j \in \omega} q_j$.

So for $(\sigma, \omega) \in I(K; [m], [m])$ and integer $k \geq 0$,

$$H_k^{\sigma, \omega}(M) = H_{k-|T(\sigma, \omega)|}^{\sigma, \omega}(K) \otimes T(\sigma, \omega), \quad H_{\sigma, \omega}^k(M) = H_{\sigma, \omega}^{k-|T(\sigma, \omega)|}(K) \otimes T(\sigma, \omega),$$

and

$$H_*(M) = \bigoplus_{(\sigma, \omega) \in I(K; [m], [m])} \Sigma^{|T(\sigma, \omega)|} H_*^{\sigma, \omega}(K), \quad H^*(M) = \bigoplus_{(\sigma, \omega) \in I(K; [m], [m])} \Sigma^{|T(\sigma, \omega)|} H_{\sigma, \omega}^*(K),$$

$$\overline{H}_*(M) = \bigoplus_{(\sigma, \omega) \in \overline{I}(K; [m], [m])} \Sigma^{|T(\sigma, \omega)|} H_*^{\sigma, \omega}(K), \quad \overline{H}^*(M) = \bigoplus_{(\sigma, \omega) \in \overline{I}(K; [m], [m])} \Sigma^{|T(\sigma, \omega)|} H_{\sigma, \omega}^*(K),$$

$$\hat{H}_*(M) = \bigoplus_{\sigma \in K} \mathbb{F}_\sigma, \quad \hat{H}^*(M) = \bigoplus_{\sigma \in K} \mathbb{F}_\sigma,$$

where \mathbb{F}_σ is a copy of \mathbb{F} regarded as a graded group with degree 0.

2 Duality

Recall that for a simplicial complex K with vertex set a subset of S , the Alexander dual of K relative to S is the simplicial complex $K^* = \{S \setminus \sigma \mid \sigma \subset S, \sigma \notin K\}$.

Theorem 2.1 *Let K be a simplicial complex with vertex set a subset of $[m]$ and K^* be the Alexander dual of K relative to $[m]$. Then for any two subsets σ, ω of $[m]$, $(\sigma, \omega) \in \overline{I}(K; [m], \Omega)$ if and only if $(\tilde{\sigma}, \omega) \in \overline{I}(K^*; [m], \Omega)$, where $\tilde{\sigma} = [m] \setminus (\sigma \cup \omega)$ and $\overline{I}(K; [m], \Omega)$ is as in Definition 1.3.*

For any $(\sigma, \omega) \in \overline{I}(K; [m], \Omega)$, there are duality isomorphisms

$$\zeta_{\sigma, \omega}(K): H_*^{\sigma, \omega}(K) \rightarrow H_{|\omega| - * - 1}^{|\omega| - * - 1}(K^*), \quad \zeta_{\tilde{\sigma}, \omega}(K^*): H_*^{\tilde{\sigma}, \omega}(K^*) \rightarrow H_{|\omega| - * - 1}^{|\omega| - * - 1}(K)$$

that are dual to each other, where $|\omega|$ is the cardinality of ω . Dually, there are duality isomorphisms

$$\zeta_{\sigma, \omega}^*(K): H_*^{\sigma, \omega}(K) \rightarrow H_{|\omega| - * - 1}^{\tilde{\sigma}, \omega}(K^*), \quad \zeta_{\tilde{\sigma}, \omega}^*(K^*): H_*^{\tilde{\sigma}, \omega}(K^*) \rightarrow H_{|\omega| - * - 1}^{\sigma, \omega}(K)$$

that are dual to each other. $\zeta_{\sigma, \omega}(K)$ and $\zeta_{\tilde{\sigma}, \omega}^*(K^*)$ are inverse of each other and $\zeta_{\sigma, \omega}^*(K)$ and $\zeta_{\tilde{\sigma}, \omega}(K^*)$ are inverse of each other.

Proof Let K^* be the Alexander dual of K relative to $[m]$ and $(K_{\sigma, \omega})^*$ be the Alexander dual of $K_{\sigma, \omega}$ relative to ω . We prove $(K_{\sigma, \omega})^* = (K^*)_{\tilde{\sigma}, \omega}$. For $\eta \in (K_{\sigma, \omega})^*$, we have $\eta \subset \omega$ and $\omega \setminus \eta \notin K_{\sigma, \omega}$. So $\sigma \cup (\omega \setminus \eta) \notin K$, $[m] \setminus (\sigma \cup (\omega \setminus \eta)) = \tilde{\sigma} \cup \eta \in K^*$, $\eta \in (K^*)_{\tilde{\sigma}, \omega}$. So $(K_{\sigma, \omega})^* \subset (K^*)_{\tilde{\sigma}, \omega}$. Conversely, for $\eta \in (K^*)_{\tilde{\sigma}, \omega}$, we have $\eta \subset \omega$ and $\tilde{\sigma} \cup \eta \in K^*$. So $[m] \setminus (\tilde{\sigma} \cup \eta) = \sigma \cup (\omega \setminus \eta) \notin K$, $\omega \setminus \eta \notin K_{\sigma, \omega}$, $\eta \in (K_{\sigma, \omega})^*$. So $(K^*)_{\tilde{\sigma}, \omega} \subset (K_{\sigma, \omega})^*$. Thus, $(K^*)_{\tilde{\sigma}, \omega} = (K_{\sigma, \omega})^*$ and the correspondence $(\sigma, \omega) \rightarrow (\tilde{\sigma}, \omega)$ is a 1-1 correspondence from $\overline{I}(K; [m], \Omega)$ to $\overline{I}(K^*; [m], \Omega)$.

Denote $K_{\tilde{\sigma}, \omega}^* = (K_{\sigma, \omega})^* = (K^*)_{\tilde{\sigma}, \omega}$. Let $(C_*(2^\omega, K_{\sigma, \omega}), d)$ be the relative simplicial chain complex. Since $\tilde{H}_*(2^\omega) = 0$, we have a boundary isomorphism $\partial: H_*(2^\omega, K_{\sigma, \omega}) \rightarrow \tilde{H}_{*-1}(K_{\sigma, \omega}) = H_*^{\sigma, \omega}(K)$. $C_*(2^\omega, K_{\sigma, \omega})$ has a base consisting of all non-simplices of $K_{\sigma, \omega}$, i.e., $K_{\sigma, \omega}^c = \{\eta \subset \omega \mid \eta \notin K_{\sigma, \omega}\}$ is a base of $C_*(2^\omega, K_{\sigma, \omega})$. The correspondence $\eta \rightarrow \omega \setminus \eta$ for all $\eta \in K_{\sigma, \omega}^c$ induces a dual complex isomorphism $\psi: (C_*(2^\omega, K_{\sigma, \omega}), d) \rightarrow (\tilde{C}^*(K_{\tilde{\sigma}, \omega}^*), \delta)$ that induces isomorphism

$\bar{\psi}: H_*(2^\omega, K_{\sigma, \omega}) \rightarrow \tilde{H}^{|\omega|-* - 2}(K_{\bar{\sigma}, \omega}^*) = H^{|\omega|-* - 1}(K^*)$, where $(\tilde{C}^*(K_{\bar{\sigma}, \omega}^*), \delta)$ is the augmented simplicial cochain complex of $K_{\bar{\sigma}, \omega}^*$. So $\zeta_{\sigma, \omega}(K) = \bar{\psi}\partial^{-1}$ is the isomorphism of the theorem.

Theorem 2.2 *Let $M = \mathcal{Z}(K; X, A)$, $(X, A) = \{(X_k, A_k)\}_{k=1}^m$, be a polyhedral product. Define its complement polyhedral product $M^c = \mathcal{Z}(K^*; X, B)$, $(X, B) = \{(X_k, B_k)\}_{k=1}^m$, where $B_k = X_k \setminus A_k$ and K^* is the Alexander dual of K relative to $[m]$. Let $\tilde{X} = X_1 \times \cdots \times X_m$. Then $M^c = \tilde{X} \setminus M$.*

Proof For a subset σ of $[m]$, 2^σ has one maximal simplex σ . If $\sigma = \{i_1, \dots, i_s\}$, then $(2^\sigma)^*$ has s maximal simplices $\bar{i}_1, \dots, \bar{i}_s$, where $\bar{i} = [m] \setminus \{i\}$. So $\tilde{X} \setminus D(\sigma) = \cup_{i \in \sigma} D^*(\bar{i})$, where D^* is defined by that

$$D^*(\omega) = Z_1 \times \cdots \times Z_m, \quad Z_i = \begin{cases} X_i & \text{if } i \in \omega, \\ B_i & \text{if } i \notin \omega. \end{cases}$$

This implies that when $K = 2^\sigma$, $\tilde{X} \setminus \mathcal{Z}(K; X, A) = \mathcal{Z}(K^*; X, B)$. It is easy to check that for any two simplicial complexes K and L , $(K \cup L)^* = K^* \cap L^*$ and $\mathcal{Z}(K^* \cap L^*; X, B) = \mathcal{Z}(K^*; X, B) \cap \mathcal{Z}(L^*; X, B)$. So for any simplicial complex K ,

$$\begin{aligned} & \mathcal{Z}(K^*; X, B) \\ &= \mathcal{Z}((\cup_{\sigma \in K} 2^\sigma)^*; X, B) = \mathcal{Z}(\cap_{\sigma \in K} (2^\sigma)^*; X, B) = \cap_{\sigma \in K} \mathcal{Z}((2^\sigma)^*; X, B) \\ &= \cap_{\sigma \in K} (\tilde{X} \setminus \mathcal{Z}(2^\sigma; X, A)) = \cap_{\sigma \in K} (\tilde{X} \setminus D(\sigma)) = \tilde{X} \setminus (\cup_{\sigma \in K} D(\sigma)) \\ &= \tilde{X} \setminus \mathcal{Z}(K; X, A). \end{aligned}$$

The theorem is proved.

By definition, the Alexander dual of $\{\phi\}$ relative to $[m]$ is $2^{[m] \setminus \{[m]\}}$, the Alexander dual of $\{\}$ relative to $[m]$ is $2^{[m]}$. So the complement polyhedral product of $\mathcal{Z}(\{\}; X, A) = \phi$ is $\mathcal{Z}(2^{[m]}; X, B) = \tilde{X}$ and the complement polyhedral product of $\mathcal{Z}(\{\phi\}; X, A) = A_1 \times \cdots \times A_m$ is $\mathcal{Z}(2^{[m] \setminus \{[m]\}}; X, B)$.

Definition 2.3 *Let $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ be a sequence of space pairs satisfying the following two conditions.*

- 1) *Every X_k is a closed orientable manifold (with respect to homology over \mathbb{F}) of dimension r_k .*
- 2) *Every A_k is a polyhedron proper subspace of X_k that is the deformation retract of a neighborhood.*

Let $(X, B) = \{(X_k, B_k)\}_{k=1}^m$ be such that $B_k = X_k \setminus A_k$ is the complement space of A_k for $k = 1, \dots, m$. We have the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A_k) & \xrightarrow{i_k} & H_n(X_k) & \xrightarrow{j_k} & H_n(X_k, A_k) & \xrightarrow{\partial_k} & H_{n-1}(A_k) & \longrightarrow & \cdots \\ & & \alpha_k \downarrow & & \gamma_k \downarrow & & \beta_k \downarrow & & \alpha_k \downarrow & & \\ \cdots & \longrightarrow & H^{r_k-n}(X_k, B_k) & \xrightarrow{q_k^*} & H^{r_k-n}(X_k) & \xrightarrow{p_k^*} & H^{r_k-n}(B_k) & \xrightarrow{\partial_k^*} & H^{r_k-n+1}(X_k, B_k) & \longrightarrow & \cdots \end{array}$$

where α_k, β_k are Alexander duality isomorphisms and γ_k is Poncaré duality isomorphism.

The duality isomorphism $\gamma_{(X, A)}: T_*(X, A) \rightarrow T^*(X, B)$ (not degree preserving!) is defined as follows. For $a_1 \otimes \cdots \otimes a_m \in$

$T_*(X, A),$

$$\gamma_{(X,A)}(a_1 \otimes \cdots \otimes a_m) = b_1 \otimes \cdots \otimes b_m, \quad b_k = \begin{cases} (\partial_k^*)^{-1}(\alpha_k(a_k)) & \text{if } a_k \in \ker i_k \\ \gamma_k(i_k(a_k)) & \text{if } a_k \in \text{coim } i_k \\ \gamma_k(a_k) & \text{if } a_k \in \text{im } i_k \\ \gamma_k(a_k) & \text{if } a_k \in \text{coker } i_k \end{cases}$$

Dually, we have the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^{n-1}(A_k) & \xrightarrow{\partial_k^*} & H^n(X_k, A_k) & \xrightarrow{j_k^*} & H^n(X_k) & \xrightarrow{i_k^*} & H^n(A_k) & \longrightarrow & \cdots \\ & & \alpha_k^* \downarrow & & \beta_k^* \downarrow & & \gamma_k^* \downarrow & & \alpha_k^* \downarrow & & \\ \cdots & \longrightarrow & H_{r_k-n+1}(X_k, B_k) & \xrightarrow{\partial_k} & H_{r_k-n}(B_k) & \xrightarrow{p_k} & H_{r_k-n}(X_k) & \xrightarrow{q_k} & H_{r_k-n}(X_k, B_k) & \longrightarrow & \cdots \end{array}$$

The duality isomorphism $\gamma_{(X,A)}^*: T^*(X, A) \rightarrow T_*(X, B)$ is defined as follows. For $a_1 \otimes \cdots \otimes a_m \in T_*(X, A),$

$$\gamma_{(X,A)}^*(a_1 \otimes \cdots \otimes a_m) = b_1 \otimes \cdots \otimes b_m, \quad b_k = \begin{cases} \partial_k(\alpha_k^*(a_k)) & \text{if } a_k \in \text{coker } i_k^* \\ \gamma_k^*((i_k^*)^{-1}(a_k)) & \text{if } a_k \in \text{im } i_k^* \\ \gamma_k^*(a_k) & \text{if } a_k \in \ker i_k^* \\ \gamma_k^*(a_k) & \text{if } a_k \in \text{coim } i_k^* \end{cases}$$

Theorem 2.4 Let everything be as in Definition 2.3. Then for any simplicial complex K with vertex set a subset of $[m], I(K; X, A) = I(K; [m], \Omega) = I(K^*; [m], \Omega) = I(K^*; X, B),$ where $\Omega = \{k \mid \ker i_k \neq 0\} = \{k \mid \ker p_k \neq 0\}.$

The duality isomorphisms $\gamma_{(X,A)}$ and $\gamma_{(X,A)}^*$ satisfy that for every $(\sigma, \omega) \in I(2^{[m]}; [m], \Omega),$ the restriction on $T_*^{\sigma, \omega}(X, A)$ and $T_{\sigma, \omega}^*(X, A)$ are the following group isomorphisms

$$\gamma_{\sigma, \omega}: T_*^{\sigma, \omega}(X, A) \rightarrow T_{\tilde{\sigma}, \omega}^{r-|\omega|-*}(X, B), \quad \gamma_{\sigma, \omega}^*: T_{\sigma, \omega}^*(X, A) \rightarrow T_{r-|\omega|-*}^{\tilde{\sigma}, \omega}(X, B),$$

where $r = r_1 + \cdots + r_m, \tilde{\sigma} = [m] \setminus (\sigma \cup \omega)$ and $|\omega|$ is the cardinality of $\omega.$ Equivalently, $\gamma_{(X,A)} = \bigoplus_{(\sigma, \omega) \in I(2^{[m]}; [m], \Omega)} \gamma_{\sigma, \omega}$ and $\gamma_{(X,A)}^* = \bigoplus_{(\sigma, \omega) \in I(2^{[m]}; [m], \Omega)} \gamma_{\sigma, \omega}^*.$

So for polyhedral product $M = \mathcal{Z}(K; X, A)$ and complement $M^c = \mathcal{Z}(K^*; X, B),$ there is a (purely algebraic) isomorphism $\rho_M: \overline{H}_*(M) \rightarrow \overline{H}^{r-*}(\overline{M}^c)$ and its dual map $\rho_M^*: \overline{H}^*(M) \rightarrow \overline{H}_{r-*}(\overline{M}^c)$ defined by

$$\rho_M = \bigoplus_{(\sigma, \omega) \in \overline{I}(K; X, A)} \zeta_{\sigma, \omega}(K) \otimes \gamma_{\sigma, \omega} \text{ and } \rho_M^* = \bigoplus_{(\sigma, \omega) \in \overline{I}(K; X, A)} \zeta_{\sigma, \omega}^*(K) \otimes \gamma_{\sigma, \omega}^*.$$

Proof From the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A_k) & \xrightarrow{i_k} & H_n(X_k) & \xrightarrow{j_k} & H_n(X_k, A_k) & \xrightarrow{\partial_k} & H_{n-1}(A_k) & \longrightarrow & \cdots \\ & & \alpha_k \downarrow & & \gamma_k \downarrow & & \beta_k \downarrow & & \gamma_k \downarrow & & \\ \cdots & \longrightarrow & H^{r_k-n}(X_k, B_k) & \xrightarrow{q_k^*} & H^{r_k-n}(X_k) & \xrightarrow{p_k^*} & H^{r_k-n}(B_k) & \xrightarrow{\partial_k^*} & H^{r_k-n+1}(X_k, B_k) & \longrightarrow & \cdots \end{array}$$

we have isomorphisms $\ker i_k \cong \text{coker } p_k^* \cong \ker p_k, \text{coker } i_k \cong \text{coim } p_k^* \cong \text{im } p_k$ and $\text{im } i_k \cong \ker p_k^* \cong \text{coker } p_k.$ So $\ker i_k \neq 0$ if and only if $\ker p_k \neq 0$ and $\{k \mid \ker i_k \neq 0\} = \{k \mid \ker p_k \neq 0\} = \Omega.$ Since $H_{r_k}(A_k) = 0, H_{r_k}(B_k) = 0, H_{r_k}(X_k) = \mathbb{F},$ we have $\text{coker } i_k \neq 0$ and $\text{coker } p_k \neq 0$ for all $k.$ So $\{k \mid \text{coker } i_k \neq 0\} = \{k \mid \text{coker } p_k \neq 0\} = [m].$

For $\gamma_{(X,A)}(a_1 \otimes \cdots \otimes a_m) = b_1 \otimes \cdots \otimes b_m,$ we have $a_k \in \ker i_k$ if and only if $b_k \in \text{coker } p_k^*;$ $a_k \in \text{im } i_k$ if and only if $b_k \in \ker p_k^*;$ $a_k \in \text{coker } i_k$ if and only if $b_k \in \text{coim } p_k^*.$ So we have the isomorphism $\gamma_{\sigma, \omega}.$

Theorem 2.5 *Let everything be as in Theorem 2.4. Then both the polyhedral product $M = \mathcal{Z}(K; X, A)$ and its complement polyhedral product $M^c = \mathcal{Z}(K^*; X, B)$ satisfy the condition of Theorem 1.6 with $I(K; X, A) = I(K; [m], \Omega) = I(K^*; [m], \Omega) = I(K^*; X, B)$. For $\tilde{X} = X_1 \times \cdots \times X_m$, we have the following long exact exact sequences,*

$$\begin{aligned} \cdots \longrightarrow H_n(M) &\xrightarrow{i_M} H_n(\tilde{X}) \xrightarrow{j_M} H_n(\tilde{X}, M) \xrightarrow{\partial_M} H_{n-1}(M) \longrightarrow \cdots \\ \cdots \longrightarrow H^{n-1}(M) &\xrightarrow{\partial_M^*} H^n(\tilde{X}, M) \xrightarrow{j_M^*} H^n(\tilde{X}) \xrightarrow{i_M^*} H^n(M) \longrightarrow \cdots \end{aligned}$$

Define $\hat{H}_*(\tilde{X}, M) = \text{im } j_M$, $\overline{H}_*(\tilde{X}, M) = \text{coim } \partial_M$, $\hat{H}^*(\tilde{X}, M) = \text{coim } j_M^*$, $\overline{H}^*(\tilde{X}, M) = \text{im } \partial_M^*$. Then

$$\begin{aligned} \overline{H}_{*+1}(\tilde{X}, M) &\cong \overline{H}_*(K; [m], \Omega) \hat{\otimes} \overline{T}_*(K; X, A), \quad \hat{H}_*(\tilde{X}, M) \cong \hat{T}_*(K^c; X, A), \\ \overline{H}^{*+1}(\tilde{X}, M) &\cong \overline{H}^*(K; [m], \Omega) \hat{\otimes} \overline{T}^*(K; X, A), \quad \hat{H}^*(\tilde{X}, M) \cong \hat{T}^*(K^c; X, A), \end{aligned}$$

where $\hat{T}_*(K^c; X, A)$, $\hat{T}^*(K^c; X, A)$ are subsystem of $T_*(X, A)$, $T^*(X, A)$ with index set $\{(\sigma, \phi) \mid \sigma \notin K\}$.

There are Alexander decomposition $H_*(M) = \hat{H}_*(M) \oplus \overline{H}_*(M)$, $H^*(M) = \hat{H}^*(M) \oplus \overline{H}^*(M)$ and complementary Alexander decomposition $H_*(\tilde{X}, M) = \hat{H}_*(\tilde{X}, M) \oplus \overline{H}_*(\tilde{X}, M)$, $H^*(\tilde{X}, M) = \hat{H}^*(\tilde{X}, M) \oplus \overline{H}^*(\tilde{X}, M)$ that satisfy the following properties. Let $\alpha_M: H_*(M) \rightarrow H^{r-*}(\tilde{X}, M^c)$ be the Alexander duality isomorphism. Then α_M is the direct sum of restrictions $\hat{\alpha}_M: \hat{H}_*(M) \rightarrow \hat{H}^{r-*}(\tilde{X}, M^c)$ and $\overline{\alpha}_M: \overline{H}_*(M) \rightarrow \overline{H}^{r-*}(\tilde{X}, M^c)$ such that $\hat{\alpha}_M = \oplus_{\sigma \in K} \gamma_{\sigma, \phi}$, $(\partial_{M^c}^*)^{-1} \overline{\alpha}_M = \rho_M$. Dually, let $\alpha_M^*: H^*(M) \rightarrow H_{r-*}(\tilde{X}, M^c)$ be the Alexander duality isomorphism. Then α_M^* is the direct sum of restrictions $\hat{\alpha}_M^*: \hat{H}^*(M) \rightarrow \hat{H}_{r-*}(\tilde{X}, M^c)$ and $\overline{\alpha}_M^*: \overline{H}^*(M) \rightarrow \overline{H}_{r-*}(\tilde{X}, M^c)$ such that $\hat{\alpha}_M^* = \oplus_{\sigma \in K} \gamma_{\sigma, \phi}^*$, $\partial_{M^c} \overline{\alpha}_M^* = \rho_M^*$. Equivalently, we have the following commutative diagrams.

$$\begin{array}{ccc} \hat{H}_*(M) & \cong & \hat{T}_*(K; X, A) \\ \hat{\alpha}_M \downarrow & & \oplus \gamma_{\sigma, \phi} \downarrow \\ \hat{H}^{r-*}(\tilde{X}, M^c) & \cong & \hat{T}^{r-*}((K^*)^c; X, B) \end{array} \quad \begin{array}{ccc} \overline{H}_*(M) & \cong & \oplus H_{\sigma, \omega}^{\sigma, \omega}(K) \otimes T_{\sigma, \omega}^{\sigma, \omega}(X, A) \\ \overline{\alpha}_M \downarrow & & \rho_M \downarrow \\ \overline{H}^{r-*}(\tilde{X}, M^c) & \xrightarrow{(\partial_{M^c}^*)^{-1}} & \oplus H_{\sigma, \omega}^{|\omega|-*} (K^*) \otimes T_{\sigma, \omega}^{r-|\omega|-*} (X, B) \end{array}$$

$$\begin{array}{ccc} \hat{H}^*(M) & \cong & \hat{T}^*(K; X, A) \\ \hat{\alpha}_M^* \downarrow & & \oplus \gamma_{\sigma, \phi}^* \downarrow \\ \hat{H}_{r-*}(\tilde{X}, M^c) & \cong & \hat{T}_{r-*}((K^*)^c; X, B) \end{array} \quad \begin{array}{ccc} \overline{H}^*(M) & \cong & \oplus H_{\sigma, \omega}^*(K) \otimes T_{\sigma, \omega}^*(X, A) \\ \overline{\alpha}_M^* \downarrow & & \rho_M^* \downarrow \\ \overline{H}_{r-*}(\tilde{X}, M^c) & \xrightarrow{\partial_{M^c}} & \oplus H_{|\omega|-*}^{\tilde{\sigma}, \omega} (K^*) \otimes T_{r-|\omega|-*}^{\tilde{\sigma}, \omega} (X, B) \end{array}$$

Proof By Theorem 1.7, $\ker i_M = \overline{H}_*(K; \Sigma, \Omega) \hat{\otimes} \overline{T}_*(K; X, A)$, $\text{coim } i_M \cong \oplus_{\sigma \in K} T_{\sigma, \omega}^{\sigma, \phi}(X, A)$. From the long exact sequence for i_M, j_M, ∂_M , $\Sigma^{-1} \text{coim } \partial_M \cong \overline{H}_*(K; [m], \Omega) \hat{\otimes} \overline{T}_*(K; X, A)$, $\text{im } j_M = \hat{T}_*(K^c; X, A)$.

For two polyhedral product $M = \mathcal{Z}(K; X, A)$ and $N = \mathcal{Z}(L; X, A)$ satisfying the condition of the theorem, we have the following commutative diagrams of Mayer-Vietoris sequences ($T = T_{\sigma, \omega}^{\sigma, \omega}(X, A)$ and $T^* = T_{\sigma, \omega}^*(X, B)$).

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_k(M \cap N) & \longrightarrow & H_k(M) \oplus H_k(N) & \longrightarrow & H_k(M \cup N) & \longrightarrow \cdots \\ & \alpha_{M \cap N} \downarrow & & \alpha_M \oplus \alpha_N \downarrow & & \alpha_{M \cup N} \downarrow & \\ \cdots \longrightarrow & H^{r-k}(\tilde{X}, (M \cap N)^c) & \longrightarrow & H^{r-k}(\tilde{X}, M^c) \oplus H^{r-k}(\tilde{X}, N^c) & \longrightarrow & H^{r-k}(\tilde{X}, (M \cup N)^c) & \longrightarrow \cdots \end{array}$$

$$\begin{array}{ccccccc} \cdots \longrightarrow & \hat{T}_k(K \cap L; X, A) & \longrightarrow & \hat{T}_k(K; X, A) \oplus \hat{T}_k(L; X, A) & \longrightarrow & \hat{T}_k(K \cup L; X, A) & \longrightarrow \cdots \\ & \hat{\alpha}_{M \cap N} \downarrow & & \hat{\alpha}_M \oplus \hat{\alpha}_N \downarrow & & \hat{\alpha}_{M \cup N} \downarrow & \\ \cdots \longrightarrow & \hat{T}^{r-k}(((K \cap L)^*)^c; X, B) & \longrightarrow & \hat{T}^{r-k}((K^*)^c; X, B) \oplus \hat{T}^{r-k}((L^*)^c; X, B) & \longrightarrow & \hat{T}^{r-k}(((K \cup L)^*)^c; X, B) & \longrightarrow \cdots \end{array}$$

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_k^{\sigma,\omega}(K \cap L) \otimes T & \longrightarrow & (H_k^{\sigma,\omega}(K) \oplus H_k^{\sigma,\omega}(L)) \otimes T & \longrightarrow & H_k^{\sigma,\omega}(K \cup L) \otimes T & \longrightarrow \cdots \\
& \zeta_{\sigma,\omega}(K \cap L) \otimes \gamma_{\sigma,\omega} \downarrow & & (\zeta_{\sigma,\omega}(K) \oplus \zeta_{\sigma,\omega}(L)) \otimes \gamma_{\sigma,\omega} \downarrow & & \zeta_{\sigma,\omega}(K \cup L) \otimes \gamma_{\sigma,\omega} \downarrow & \\
\cdots \longrightarrow & H_{\bar{\sigma},\omega}^{|\omega|-k-1}((K \cap L)^*) \otimes T^* & \longrightarrow & (H_{\bar{\sigma},\omega}^{|\omega|-k-1}(K^*) \oplus H_{\bar{\sigma},\omega}^{|\omega|-k-1}(L^*)) \otimes T^* & \longrightarrow & H_{\bar{\sigma},\omega}^{|\omega|-k-1}((K \cup L)^*) \otimes T^* & \longrightarrow \cdots
\end{array}$$

This implies that if Alexander and complementary Alexander decompositions exist for M, N and $M \cap N$, then they exist for $M \cup N$. So by induction on the number of maximal simplices, we need only prove that Alexander and complementary Alexander decompositions exist for $M = \mathcal{Z}(K; X, A)$ such that K has only one maximal simplex. This is checked in the following Example 2.6.

Example 2.6 Let $M = \mathcal{Z}(K; X, A)$ be a polyhedral product satisfying the condition of Theorem 2.5 such that K has only one maximal simplex, i.e., $M = \mathcal{Z}(2^\tau; X, A) = Y_1 \times \cdots \times Y_m$, where $Y_k = X_k$ if $k \in \tau$ and $Y_k = A_k$ if $k \notin \tau$. Then $K^* = 2^\tau * (2^{\bar{\tau}} \setminus \{\bar{\tau}\})$, where $\bar{\tau} = [m] \setminus \tau$ and $M^c = \mathcal{Z}(K^*; X, B)$ with $B_k = X_k \setminus A_k$.

By Theorem 1.6, $H_*(M) = H_*(2^\tau; \Sigma, \Omega) \hat{\otimes} T_*(2^\tau; X, A)$. For $(\sigma, \omega) \in I(2^\tau; X, A)$, $K_{\sigma,\omega} = \{\phi\}$ if $\omega \cap \tau = \phi$ and $K_{\sigma,\omega} = 2^{\omega \cap \tau}$ if $\omega \cap \tau \neq \phi$. So $H_*^{\sigma,\omega}(K) = \mathbb{F}$ if $\omega \cap \tau = \phi$ and $H_*^{\sigma,\omega}(K) = 0$ if $\omega \cap \tau \neq \phi$. So by identifying $\mathbb{F} \otimes T_*^{\sigma,\omega}(M)$ with $T_*^{\sigma,\omega}(M)$, we have that $H_*(M) = T_*(2^\tau; X, A) = \hat{T}_*(2^\tau; X, A) \oplus \bar{T}_*(2^\tau; X, A)$. We have Künneth isomorphism $H_*(M) = H_*(Y_1) \otimes \cdots \otimes H_*(Y_m)$. Since both $H_*(A_k)$ and $H_*(X_k)$ are subgroup of $H_*(A_k) \cup_{i_k} H_*(X_k)$, $H_*(Y_1) \otimes \cdots \otimes H_*(Y_m)$ is a subgroup of $T_*(X, A)$. Take the Alexander decomposition of $H_*(M)$ to be $\hat{H}_*(M) = \hat{T}_*(2^\tau; X, A) = \{a_1 \otimes \cdots \otimes a_m \in H_*(Y_1) \otimes \cdots \otimes H_*(Y_m) \mid \text{there is no } a_k \in \ker i_k\}$ and $\bar{H}_*(M) = \bar{T}_*(2^\tau; X, A) = \{a_1 \otimes \cdots \otimes a_m \in H_*(Y_1) \otimes \cdots \otimes H_*(Y_m) \mid \text{there is } a_k \in \ker i_k\}$.

We have space pair equality $(\tilde{X}, M^c) = (X_1, Z_1) \times \cdots \times (X_m, Z_m)$, where $Z_k = \phi$ if $k \in \tau$ and $Z_k = B_k$ if $k \notin \tau$. By Künneth theorem, $H^*(\tilde{X}, M^c) = H^*(X_1, Z_1) \otimes \cdots \otimes H^*(X_m, Z_m)$. From the following commutative diagram of isomorphisms (ρ_k is Poncaré or Alexander duality isomorphism)

$$\begin{array}{ccc}
H_*(M) & \cong & H_*(Y_1) \otimes \cdots \otimes H_*(Y_m) \\
\alpha_M \downarrow & & \rho_1 \otimes \cdots \otimes \rho_m \downarrow \\
H^*(\tilde{X}, M^c) & \cong & H^*(X_1, Z_1) \otimes \cdots \otimes H^*(X_m, Z_m)
\end{array}$$

we have the complementary Alexander decomposition

$$\begin{aligned}
\hat{H}^*(\tilde{X}, M^c) &= \{a_1 \otimes \cdots \otimes a_m \in H^*(X_1, Z_1) \otimes \cdots \otimes H^*(X_m, Z_m) \mid \text{there is no } a_k \in \text{im } \partial_k^*\}, \\
\bar{H}^*(\tilde{X}, M^c) &= \{a_1 \otimes \cdots \otimes a_m \in H^*(X_1, Z_1) \otimes \cdots \otimes H^*(X_m, Z_m) \mid \text{there is } a_k \in \text{im } \partial_k^*\}.
\end{aligned}$$

For $x = a_1 \otimes \cdots \otimes a_m \in \hat{T}_*(2^\tau; X, A)$, $j_{M^c}^*(\hat{\alpha}_M(x)) = b_1 \otimes \cdots \otimes b_m$, where $b_k = \gamma_k(a_k)$ if $k \in \tau$; $b_k = \gamma_k(i_k(a_k))$ if $k \notin \tau$ and $a_i \in \text{coim } i_k$. So $j_{M^c}^*(\hat{\alpha}_M(x)) = \gamma_{(X,A)}(x)$. $x \in T_*^{\sigma,\phi}(X, A)$ if and only if $j_{M^c}^*(\hat{\alpha}_M(x)) \in T_{\bar{\sigma},\phi}^*(X, B)$ ($\bar{\sigma} = [m] \setminus \sigma$). So by identifying $\hat{H}^*(\tilde{X}, M^c)$ with $\oplus_{\sigma \in K} T_{\bar{\sigma},\phi}^*(X, B) = T^*((K^*)^c; X, B)$, we have $\hat{\alpha}_M = \oplus_{\sigma \in K} \gamma_{\sigma,\phi}$.

Now we determine the subgroup $\bar{H}^*(M^c)$ of $H^*(M^c)$. For $(\sigma, \omega) \in I(K^*; X, B)$, $K_{\sigma,\omega}^* = 2^\omega \setminus \{\omega\}$ if $\omega \cap \tau = \phi$ and $K_{\sigma,\omega}^* = 2^{\omega'} * (2^{\omega''} \setminus \{\omega''\})$ if $\omega \cap \tau \neq \phi$, where $\omega' = \omega \cap \tau$ and $\omega'' = \omega \setminus \tau$. Notice that the geometrical realization of $2^\omega \setminus \{\omega\}$ is a $(|\omega|-2)$ -dimensional sphere. So $H_{\sigma,\phi}^*(K^*) = \mathbb{F}$; $H_{\sigma,\omega}^*(K^*) = \Sigma^{|\omega|-1} \mathbb{F}$ if $\omega \cap \tau = \phi$ and $\omega \neq \phi$;

$H_{\sigma,\omega}^*(K^*) = 0$ if $\omega \cap \tau \neq \phi$. Take $\overline{H}^*(M^c) = \bigoplus_{(\sigma,\omega) \in \overline{I}(K^*;[m],\Omega), \omega \cap \tau = \phi} H_{\sigma,\omega}^*(K^*) \otimes T_{\sigma,\omega}^*(X, B)$. Since $\zeta_{\sigma,\omega}(K): H_{\sigma,\omega}^*(K) \rightarrow H_{\tilde{\sigma},\omega}^{|\omega|-* -1}(K^*)$ is an isomorphism between 1-dimensional spaces, we may identify $\zeta_{\sigma,\omega}(K) \otimes \gamma_{\sigma,\omega}: H_{\sigma,\omega}^*(K) \otimes T_{\sigma,\omega}^*(X, A) \rightarrow H_{\tilde{\sigma},\omega}^{|\omega|-* -1}(K^*) \otimes T_{\tilde{\sigma},\omega}^{r-|\omega|-*}(X, B)$ with $\Sigma^{|\omega|-1} \gamma_{\sigma,\omega}: T_{\sigma,\omega}^{\sigma,\omega}(X, A) \rightarrow T_{\tilde{\sigma},\omega}^{r-* -1}(X, B)$.

Define $\lambda: \overline{H}^*(M^c) \rightarrow H^*(\tilde{X}, M^c)$ as follows. For $x = a_1 \otimes \cdots \otimes a_m \in \overline{H}^*(M^c)$, $\lambda(x) = b_1 \otimes \cdots \otimes b_m$, where

$$b_k = \begin{cases} a_k & \text{if } k \in \tau, \\ \partial_k^*(a_k) & \text{if } a_k \in \text{coker } p_k^* \text{ and } k \notin \tau, \\ (q_k^*)^{-1}(a_k) & \text{if } a_k \in \ker p_k^* \text{ and } k \notin \tau. \end{cases}$$

Then $j_{M^c}^*(\lambda(x)) = q_1^*(b_1) \otimes \cdots \otimes q_m^*(b_m) = 0$, since there is $a_k \in \text{coker } p_k^*$ and $q_k^*(\partial_k^*(a_k)) = 0$. So λ is just the restriction of $\partial_{M^c}^*$ on $\overline{H}^*(M^c)$. It is easy to check that for $x = a_1 \otimes \cdots \otimes a_m \in \overline{T}_*(2^\tau; X, A)$, $\lambda \gamma_{\sigma,\omega}(x) = \rho_1(a_1) \otimes \cdots \otimes \rho_m(a_m) = \overline{\alpha}_M(x)$. So $(\partial_{M^c}^*)^{-1} \overline{\alpha}_M = \bigoplus_{(\sigma,\omega) \in \overline{I}(K;[m],\Omega)} \Sigma^{|\omega|-1} \gamma_{\sigma,\omega} = \bigoplus_{(\sigma,\omega) \in \overline{I}(K;[m],\Omega)} \zeta_{\sigma,\omega}(K) \otimes \gamma_{\sigma,\omega} = \rho_M$.

Example 2.7 Let $M = \mathcal{Z}_K \left(\begin{smallmatrix} r_1+1 & \cdots & r_m+1 \\ q_1 & \cdots & q_m \end{smallmatrix} \right)$ be as in Example 1.8. Since S^{r-q} is a deformation retract of $S^{r+1} - \theta(S^q)$, M^c is homotopic equivalent to $\mathcal{Z}_{K^*} \left(\begin{smallmatrix} r_1+1 & \cdots & r_m+1 \\ r_1-q_1 & \cdots & r_m-q_m \end{smallmatrix} \right)$. By the computation in 1.8,

$$\overline{H}_*(M) = \bigoplus_{(\sigma,\omega) \in \overline{I}(K;X,A)} \Sigma^{|T(\sigma,\omega)|} H_{\sigma,\omega}^{\sigma,\omega}(K), \quad \overline{H}^*(M^c) = \bigoplus_{(\sigma',\omega) \in \overline{I}(K^*;X,B)} \Sigma^{|T(\sigma',\omega)|} H_{\sigma',\omega}^*(K^*).$$

Since $H_{\sigma,\omega}^{\sigma,\omega}(K) \cong H_{\tilde{\sigma},\omega}^{|\omega|-* -1}(K^*)$ and $|T(\sigma,\omega)| + |T(\tilde{\sigma},\omega)| = r - |\omega|$ ($\tilde{\sigma} = [m] \setminus (\sigma \cup \omega)$), $\overline{H}_*(M) \cong \overline{H}^{r-* -1}(M^c)$.

Specifically, denote $\mathcal{Z}(K; r) = \mathcal{Z}_K \left(\begin{smallmatrix} 2r+1 & \cdots & 2r+1 \\ r & \cdots & r \end{smallmatrix} \right)$. Then we have $\overline{H}_*(\mathcal{Z}(K; r)) \cong \overline{H}^{(2r+1)m-* -1}(\mathcal{Z}(K^*; r))$.

3 The cohomology algebra of polyhedral products

Convention In the later part of this paper, a coalgebra (A, Δ) is a pair such that $\Delta: A \rightarrow A \otimes A$ is a graded group homomorphism. Dually, an algebra (A, Π) is a pair such that $\Pi: A \otimes A \rightarrow A$ is a graded group homomorphism. (Co)associativity and unit condition is not required for a (co)algebra.

To simplify notation, we always use the same symbol to denote a chain complex homomorphism and the homology group homomorphism induced by it.

Definition 3.1 Let everything be as in Definition 1.3.

The homology coproduct system of K with index set $I(K; \Sigma, \Omega)$ is a pair $(H_*(K; \Sigma, \Omega), \Delta^R)$ defined as follows. For every three set pairs $(\sigma, \omega), (\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$ such that $\sigma' \cup \sigma'' \subset \sigma, \omega \subset \omega' \cup \omega''$, there is a restricted coproduct

$$\Delta^R: H_{\sigma,\omega}^{\sigma,\omega}(K) \rightarrow H_{\sigma',\omega'}^{\sigma',\omega'}(K) \otimes H_{\sigma'',\omega''}^{\sigma'',\omega''}(K)$$

induced by the chain complex homomorphism $\Delta^R: \Sigma \tilde{C}_*(K_{\sigma,\omega}) \rightarrow \Sigma \tilde{C}_*(K_{\sigma',\omega'}) \otimes \Sigma \tilde{C}_*(K_{\sigma'',\omega'')}$ ($\Sigma \tilde{C}_*$ is the augmented simplicial chain complex with degree uplifted by 1) defined as follows. For an ordered simplex $\eta \in K_{\sigma,\omega}$, $\Delta^R(\eta) = (-1)^\tau \eta' \otimes \eta''$, where η', η'' are respectively ordered simplices of $K_{\sigma',\omega'}, K_{\sigma'',\omega''}$ such that $\eta' = \eta \cap \omega', \eta'' = \eta \setminus \eta'$ and $(-1)^\tau$ is the sign of the permutation from η to $\eta' \cup \eta''$. Precisely, for $\{i_1, \dots, i_s\} \in K_{\sigma,\omega}$ with $i_1 < \cdots < i_s$,

$$\Delta^R(\{i_1, \dots, i_s\}) = (-1)^\tau \{j_1, \dots, j_u\} \otimes \{k_{u+1}, \dots, k_s\},$$

where $j_1 < \dots < j_u$, $k_{u+1} < \dots < k_s$, $\{j_1, \dots, j_u\} = \{i_1, \dots, i_s\} \cap \omega'$, $\{j_1, \dots, j_u\} \cup \{k_{u+1}, \dots, k_s\} = \{i_1, \dots, i_s\}$, $(-1)^\tau$ is the sign of the permutation $\begin{pmatrix} i_1 & \dots & i_u & i_{u+1} & \dots & i_s \\ j_1 & \dots & j_u & k_{u+1} & \dots & k_s \end{pmatrix}$. It is easy to check that $(d \otimes d) \Delta^R = \Delta^R d$.

Dually, the cohomology product system of K with index set $I(K; \Sigma, \Omega)$ is a pair $(H^*(K; \Sigma, \Omega), \Pi_R)$ defined as follows. For every three set pairs $(\sigma, \omega), (\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$ such that $\sigma' \cup \sigma'' \subset \sigma$, $\omega \subset \omega' \cup \omega''$, there is a restricted product

$$\Pi_R: H_{\sigma', \omega'}^*(K) \otimes H_{\sigma'', \omega''}^*(K) \rightarrow H_{\sigma, \omega}^*(K)$$

that is the dual map of $\Delta^R: H_*^{\sigma, \omega}(K) \rightarrow H_*^{\sigma', \omega'}(K) \otimes H_*^{\sigma'', \omega''}(K)$.

Remark The restricted coproduct Δ^R is the core of the cohomology algebra of polyhedral product. $\Delta^R: \Sigma \tilde{C}_*(K) \rightarrow \Sigma \tilde{C}_*(K') \otimes \Sigma \tilde{C}_*(K'')$ is defined for all simplicial complexes K, K', K'' such that the vertex set of K is a subset of the union of the vertex set of K' and K'' . Specifically, if K is a subcomplex of $K' \cup K''$, then we can prove that the restricted coproduct is a simplicial approximation of the geometrical diagonal map $\Delta: (|2^S|, |K|) \rightarrow (|2^S|, |K'|) \times (|2^S|, |K''|) = (|2^S| \times |2^S|, (|K'| \times |2^S|) \cup (|2^S| \times |K''|))$ defined by $\Delta(x) = (x, x)$, where $|\cdot|$ is the geometrical realization space and S is any set containing all the vertices of the three simplicial complexes. But in general, the restricted coproduct is purely combinatorial.

Notice that when tensor product is involved, the gradation of $\Sigma \tilde{C}_*(K)$ is natural for the restricted coproduct, i.e., a simplex $\{i_1, \dots, i_s\}$ as a base element of $\Sigma \tilde{C}_*(K)$ has degree s .

Definition 3.2 Let $i: (A, \Delta^A) \rightarrow (B, \Delta^B)$ be a coalgebra homomorphism, i.e., $(i \otimes i) \Delta^A = \Delta^B i$. By Definition 1.4, a decomposition $A = \ker i \oplus \mathfrak{i}$ determines a group $A \cup_i B$. Take a subgroup \mathfrak{c} of B such that $B = \text{im } i \oplus \mathfrak{c}$ (so $\mathfrak{c} \cong \text{coker } i$). The coalgebra $(A \cup_i B, \Delta)$ is defined by $\Delta|_A = \Delta^A$ and $\Delta|_{\mathfrak{c}} = \Delta^B$. It is obvious that (A, Δ^A) is a subcoalgebra of $(A \cup_i B, \Delta)$ and $\ker i$ is a coideal of $A \cup_i B$ such that the quotient coalgebra $((A \cup_i B)/\ker i, \tilde{\Delta}) \cong (B, \Delta^B)$.

Dually, let $i: (B, \Pi_B) \rightarrow (A, \Pi_A)$ be an algebra homomorphism, i.e., $\Pi_A(i \otimes i) = i \Pi_B$. The algebra $(B \cup_i A, \Pi)$ is defined by $(B \cup_i A, \Pi) = (A^* \cup_i B^*, \Pi^*)^*$, where $*$ means the dual object. Then (B, Π_B) is a subalgebra of $(B \cup_i A, \Pi)$ and $\ker i$ is an ideal of $B \cup_i A$ such that the quotient algebra $((B \cup_i A)/\ker i, \tilde{\Pi}) \cong (A, \Pi_A)$.

Notice that the coalgebra $(A \cup_i B, \Delta)$ depends on the choice of the subgroup \mathfrak{i} and \mathfrak{c} . So $(A \cup_i B, \Delta)$ may not be coassociative even if both (A, Δ^A) and (B, Δ^B) are coassociative. The algebra case is similar.

The following theorem needs the definition of tensor product of coalgebras. Recall that for coalgebras (A_i, Δ_i) , $i = 1, 2$, the tensor product coalgebra $(A_1 \otimes A_2, \Delta_1 \otimes \Delta_2)$ is defined as follows. For $a_i \in A_i$ with $\Delta_i(a_i) = \Sigma a'_i \otimes a''_i$, $(\Delta_1 \otimes \Delta_2)(a_1 \otimes a_2) = \Sigma (-1)^{|a'_1| |a'_2|} (a'_1 \otimes a'_2) \otimes (a''_1 \otimes a''_2)$.

Theorem 3.3 Let everything be as in Definition 1.5. The homology generating coalgebra of (X, A) is defined by

$$(T_*(X, A), \Delta^T) \cong ((H_*(A_1) \cup_{i_1} H_*(X_1)) \otimes \cdots \otimes (H_*(A_m) \cup_{i_m} H_*(X_m)), \Delta_1 \otimes \cdots \otimes \Delta_m),$$

where Δ_k is the coproduct of $H_*(A_k) \cup_{i_k} H_*(X_k)$.

Then for a simplicial complex K with vertex set a subset of $[m]$, the homology generating system $T_*(K; X, A)$ is a subcoalgebra of $(T_*(X, A), \Delta^T)$ and is called the homology generating coalgebra of K with respect to (X, A) .

Dually, the cohomology generating algebra of (X, A) is defined by

$$(T^*(X, A), \Pi_T) \cong ((H^*(X_1) \cup_{i_1^*} H^*(A_1)) \otimes \cdots \otimes (H^*(X_m) \cup_{i_m^*} H^*(A_m)), \Pi_1 \otimes \cdots \otimes \Pi_m),$$

where Π_k is the product of $H^*(X_k) \cup_{i_k^*} H^*(A_k)$.

Then for a simplicial complex K with vertex set a subset of $[m]$, the cohomology generating system $T^*(K; X, A)$ is a subalgebra of $(T^*(X, A), \Pi_T)$ and is called the cohomology generating algebra of K with respect to (X, A) .

Proof Let $x = a_1 \otimes \cdots \otimes a_m \in T_*^{\sigma, \omega}(K; X, A)$, i.e., $a_k \in \text{coker } i_k$ if $k \in \sigma$; $a_k \in \ker i_k$ if $k \in \omega$; $a_k \in \text{coim } i_k$ otherwise. Suppose $\Delta_k(a_k) = \Sigma a'_k \otimes a''_k$. Then $\Delta^T(x) = \Sigma \pm (a'_1 \otimes \cdots \otimes a'_m) \otimes (a''_1 \otimes \cdots \otimes a''_m)$. If $a_k \in \text{coker } i_k$, then $a'_k, a''_k \in \text{coker } i_k$ or $\text{im } i_k$. If $a_k \in \text{coim } i_k$, then $a'_k, a''_k \in \text{coim } i_k$ or $\ker i_k$. If $a_k \in \ker i_k$, then either both a'_k and a''_k are in $\ker i_k$, or one of them is in $\text{coim } i_k$ and the other is in $\ker i_k$. This implies that if $a'_1 \otimes \cdots \otimes a'_m \in T_*^{\sigma', \omega'}(X, A)$ and $a''_1 \otimes \cdots \otimes a''_m \in T_*^{\sigma'', \omega''}(X, A)$, then $\sigma' \subset \sigma$, $\sigma'' \subset \sigma$ and $\omega \subset \omega' \cup \omega''$. Since K is a simplicial complex, $\sigma \in K$ implies $\sigma', \sigma'' \in K$. So $T_*^{\sigma', \omega'}(X, A), T_*^{\sigma'', \omega''}(X, A) \in T_*(K; X, A)$. $T_*(K; X, A)$ is a subcoalgebra of $T_*(X, A)$.

Theorem 3.4 Let everything be as in Theorem 1.6.

The coproduct Δ of $H_*(M) = H_*(K; \Sigma, \Omega) \widehat{\otimes} T_*(K; X, A)$ is as follows. Let $a \in H_*^{\sigma, \omega}(K)$ and $x \in T_*^{\sigma, \omega}(X, A)$. If $\Delta^T(x) = \Sigma_i x'_i \otimes x''_i$ with $x'_i \in T_*^{\sigma'_i, \omega'_i}(X, A)$ and $x''_i \in T_*^{\sigma''_i, \omega''_i}(X, A)$, then we denote by Δ_i the restricted coproduct $\Delta^R: H_*^{\sigma, \omega}(K) \rightarrow H_*^{\sigma'_i, \omega'_i}(K) \otimes H_*^{\sigma''_i, \omega''_i}(K)$ defined in Definition 3.1. Suppose $\Delta_i(a) = \Sigma_j a'_{i,j} \otimes a''_{i,j}$. then

$$\Delta(a \otimes x) = \Sigma_{i,j} (-1)^{|x'_i| |a''_{i,j}|} (a'_{i,j} \otimes x'_i) \otimes (a''_{i,j} \otimes x''_i).$$

Dually, the cup product Π of $H^*(M) = H^*(K; \Sigma, \Omega) \widehat{\otimes} T^*(K; X, A)$ is as follows. Let $a' \in H_*^{\sigma', \omega'}(K)$, $a'' \in H_*^{\sigma'', \omega''}(K)$, $x' \in T_*^{\sigma', \omega'}(X, A)$ and $x'' \in T_*^{\sigma'', \omega''}(X, A)$. If $\Pi_T(x', x'') = \Sigma_k y_k$ with $y_k \in T_{\sigma_k, \omega_k}^*(X, A)$, then we denote by Π_k the restricted product $\Pi_R: H_*^{\sigma', \omega'}(K) \otimes H_*^{\sigma'', \omega''}(K) \rightarrow H_{\sigma_k, \omega_k}^*(K)$ defined in Definition 3.1. Suppose $\Pi_k(a', a'') = \Sigma_l b_{k,l}$. Then

$$(a' \otimes x') \cup (a'' \otimes x'') = \Pi(a' \otimes x', a'' \otimes x'') = (-1)^{|x'| |a''|} \Sigma_{k,l} b_{k,l} \otimes y_k.$$

Proof In the following proof, we use Φ to denote all the isomorphisms switching the factors of a tensor product graded group. Precisely, let (i_1, \dots, i_n) be a permutation of $(1, \dots, n)$, then $\Phi: A_1 \otimes \cdots \otimes A_n \rightarrow A_{i_1} \otimes \cdots \otimes A_{i_n}$. The exact formula for Φ is quite complicated, but the regulation is simple. $\Phi(a_1 \otimes \cdots \otimes a_m) = (-1)^\tau a_{i_1} \otimes \cdots \otimes a_{i_n}$, where the sign is determined by identifying $\cdots \otimes x \otimes y \otimes \cdots$ with $(-1)^{|x||y|} \cdots \otimes y \otimes x \otimes \cdots$. So for coalgebras (A_k, Δ^k) , $k = 1, \dots, n$, the coproduct of the tensor product coalgebra $A_1 \otimes \cdots \otimes A_n$ is $\Delta^{(n)}: A_1 \otimes \cdots \otimes A_n \xrightarrow{\Delta^1 \otimes \cdots \otimes \Delta^n} (A_1 \otimes A_1) \otimes \cdots \otimes (A_n \otimes A_n) \xrightarrow{\Phi}$

$(A_1 \otimes \cdots \otimes A_n) \otimes (A_1 \otimes \cdots \otimes A_n)$. $\Delta^{(n)}$ is simply denoted by $\Delta^1 \otimes \cdots \otimes \Delta^n$ in other part of the paper (for example, Theorem 3.3).

Let everything be as in the proof of Theorem 1.6. Specifically, the base of $\ker i_k$, $\text{coim } i_k = \text{im } i_k$ and $\text{coker } i_k$ are respectively \mathfrak{k}_k , \mathfrak{i}_k and \mathfrak{c}_k and d is a 1-1 correspondence from \mathfrak{q}_k to \mathfrak{k}_k . Then $\mathfrak{k}_k \cup \mathfrak{i}_k \cup \mathfrak{c}_k$ is a base of $H_*(A_k) \cup_{i_k} H_*(X_k)$ and so $H_*(A_k) \cup_{i_k} H_*(X_k)$ is a subgroup of $U(k)$. Define chain complex homomorphism $\Delta_k: U(k) \rightarrow U(k) \otimes U(k)$ that is an extension of the coproduct of $H_*(A_k) \cup_{i_k} H_*(X_k)$ as follows. For $y \in \mathfrak{q}_k$ such that $dy = x$ and $\Delta_k(x) = \sum_i x'_i \otimes x''_i$, define $\Delta_k(y) = \sum_i \bar{x}'_i \otimes \bar{x}''_i$, where $d\bar{x}'_i = x'_i$, $\bar{x}''_i = x''_i$ if $x'_i \in \mathfrak{k}_k$ and $\bar{x}'_i = x'_i$, $d\bar{x}''_i = (-1)^{|x'|} x''_i$ if $x'_i \notin \mathfrak{k}_k$. Δ_k is neither coassociative nor cocommutative but satisfies $(d \otimes d)\Delta_k = \Delta_k d$. Let $\bar{\Delta}_k: S_*(X_k) \rightarrow S_*(X_k) \otimes S_*(X_k)$ be the coproduct induced by the geometrical diagonal map $\Delta(x) = (x, x)$. $U_*(k)$ is a chain subcomplex of $S_*(X_k)$ but Δ_k may not be the restriction of $\bar{\Delta}_k$ on $U_*(k)$. Define chain homotopy ϑ_k from $\bar{\Delta}_k$ to Δ_k as follows. For $a \in \mathfrak{i}_k \cup \mathfrak{c}_k$, since $(\bar{\Delta}_k - \Delta_k)(a) \sim 0$, we may define $\vartheta_k(a)$ to be any fixed element such that $(d \otimes d)(\vartheta_k(a)) = (\bar{\Delta}_k - \Delta_k)(a)$. For $a \in \mathfrak{k}_k$, $b \in \mathfrak{q}_k$, $db = a$, we have $(d \otimes d)(\bar{\Delta}_k - \Delta_k)(b) = (\bar{\Delta}_k - \Delta_k)(a)$. Define $\vartheta_k(a) = (\bar{\Delta}_k - \Delta_k)(b)$ and $\vartheta_k(b) = 0$. It is obvious that $\bar{\Delta}_k - \Delta_k = \vartheta_k d + (d \otimes d)\vartheta_k$ and so the following diagrams are homotopy commutative.

$$\begin{array}{ccccccc} (V_*(k), d) & \xrightarrow{i_k} & (S_*(A_k), d) & & (U_*(k), d) & \xrightarrow{i_k} & (S_*(X_k), d) \\ \Delta_k \downarrow & & \bar{\Delta}_k \downarrow & & \Delta_k \downarrow & & \bar{\Delta}_k \downarrow \\ ((V_*(k) \otimes (V_*(k), d \otimes d)) & \xrightarrow{i_k \otimes i_k} & (S_*(A_k) \otimes S_*(A_k), d \otimes d) & & (U_*(k) \otimes (U_*(k), d \otimes d)) & \xrightarrow{i_k \otimes i_k} & (S_*(X_k) \otimes S_*(X_k), d \otimes d) \end{array}$$

Let $(C_*(m), d) = (U_*(1) \otimes \cdots \otimes U_*(m), d \otimes \cdots \otimes d)$ and $(S_*(m), d) = (S_*(X_1) \otimes \cdots \otimes S_*(X_m), d \otimes \cdots \otimes d)$. The coproduct chain complex homomorphisms are as follows.

$$\begin{aligned} \Delta^{(m)}: C_*(m) &\xrightarrow{\Delta_1 \otimes \cdots \otimes \Delta_m} U_*(1) \otimes U_*(1) \otimes \cdots \otimes U_*(m) \otimes U_*(m) \xrightarrow{\Phi} C_*(m) \otimes C_*(m), \\ \bar{\Delta}^{(m)}: S(m) &\xrightarrow{\bar{\Delta}_1 \otimes \cdots \otimes \bar{\Delta}_m} S_*(X_1) \otimes S_*(X_1) \otimes \cdots \otimes S_*(X_m) \otimes S_*(X_m) \xrightarrow{\Phi} S_*(m) \otimes S_*(m). \end{aligned}$$

The homotopy $\vartheta^{(m)}$ from $\bar{\Delta}^{(m)}$ to $\Delta^{(m)}$ is defined as follows. Let $\bar{\vartheta}^{(m)} = \bigoplus_{k=1}^m \bar{\Delta}_1 \otimes \cdots \otimes \bar{\Delta}_{k-1} \otimes \vartheta_k \otimes \Delta_{k+1} \otimes \cdots \otimes \Delta_m$, then $\vartheta^{(m)} = \Phi \bar{\vartheta}^{(m)}: C_*(m) \xrightarrow{\bar{\vartheta}^{(m)}} S_*(X_1) \otimes S_*(X_1) \otimes \cdots \otimes S_*(X_m) \otimes S_*(X_m) \xrightarrow{\Phi} S_*(m) \otimes S_*(m)$. It is easy to check that $(d \otimes d)\vartheta^{(m)} + \vartheta^{(m)}d = \bar{\Delta}^{(m)} - \Delta^{(m)}$.

Similar to the construction of $(C_*(M), d)$, define chain subcomplex $(\bar{S}_*(M), d)$ of $(S_*(M), d)$ as follows. For a subset $\sigma \subset [m]$, define $(\bar{S}_*(\sigma), d) = (T_*(1) \otimes \cdots \otimes T_*(m), d \otimes \cdots \otimes d)$, where $T_*(k) = S_*(X_k)$ if $k \in \sigma$ and $T_*(k) = S_*(A_k)$ if $k \notin \sigma$. Then for $\sigma \subset \sigma'$, $(\bar{S}_*(\sigma), d)$ is a chain subcomplex of $(\bar{S}_*(\sigma'), d)$ and for all $\sigma, \tau \subset [m]$, $\bar{S}_*(\sigma) \cap \bar{S}_*(\tau) = \bar{S}_*(\sigma \cap \tau)$. Define $\bar{S}_*(M) = \bigoplus_{\sigma \in K} \bar{S}_*(\sigma)$. Then $(\bar{S}_*(M), d)$ is a chain subcomplex of $(S_*(M), d)$ and denote the inclusion by $\bar{i}_M: (\bar{S}_*(M), d) \rightarrow (S_*(M), d)$. Similar to the proof for i_M , we have that \bar{i}_M is a chain homotopy equivalence. By definition, $\bar{\Delta}^{(m)}$ is the restriction of $\Delta: S_*(\tilde{X}) \rightarrow S_*(\tilde{X}) \otimes S_*(\tilde{X})$ ($\tilde{X} = X_1 \times \cdots \times X_m$ and Δ is the approximation of the geometrical diagonal map $\Delta(x) = (x, x)$). So $\bar{S}_*(M)$ is a subcoalgebra of $S_*(M)$. Denote the restriction of $\bar{\Delta}^{(m)}$ on $\bar{S}_*(M)$ by $\bar{\Delta}^M: \bar{S}_*(M) \rightarrow \bar{S}_*(M) \otimes \bar{S}_*(M)$. Let $i'_M: C_*(M) \rightarrow \bar{S}_*(M)$ be the inclusion. Then we have the following (not commutative!) diagram

$$\begin{array}{ccccc}
C_*(M) & \xrightarrow{i'_M} & \overline{S}_*(M) & \xrightarrow{\overline{i}_M} & S_*(M) \\
\Delta^M \downarrow & & \overline{\Delta}^M \downarrow & & \Delta \downarrow \\
C_*(M) \otimes C_*(M) & \xrightarrow{i'_M \otimes i'_M} & \overline{S}_*(M) \otimes \overline{S}_*(M) & \xrightarrow{\overline{i}_M \otimes \overline{i}_M} & S_*(M) \otimes S_*(M)
\end{array}$$

such that $\Delta \overline{i}_M = (\overline{i}_M \otimes \overline{i}_M) \overline{\Delta}^M$ but $\overline{\Delta}^M i'_M$ and $(i'_M \otimes i'_M) \Delta^M$ may not be equal. However, the restriction of $\vartheta^{(m)}$ on $C_*(M)$ is a homotopy from $\overline{\Delta}^M i'_M$ to $(i'_M \otimes i'_M) \Delta^M$. So we have the following commutative diagram of homology groups

$$\begin{array}{ccc}
H_*(C_*(M), d) & \xrightarrow{i_M} & H_*(M) \\
\Delta^M \downarrow & & \Delta \downarrow \\
H_*(C_*(M), d) \otimes H_*(C_*(M), d) & \xrightarrow{i_M \otimes i_M} & H_*(M) \otimes H_*(M).
\end{array}$$

Now we compute the coproduct of $H_*(M)$ by Δ^M . The generating coalgebra $(T_*(K; X, A), \Delta^T)$ is a subcoalgebra of $(C_*(M), \Delta^M)$. From the proof of Theorem 1.6, we have the direct sum decomposition $(C_*(M), d) = \oplus_{x \in \Lambda} (C_*(x), d)$, where Λ is the base of $T_*(K; X, A)$. For $x \in \Lambda$, if $\Delta^T(x) = \sum_i c_i x'_i \otimes x''_i$, $x'_i, x''_i \in \Lambda$, $c_i \in \mathbb{F} \setminus \{0\}$, $x'_i \in T_*^{\sigma'_i, \omega'_i}(X, A)$, $x''_i \in T_*^{\sigma''_i, \omega''_i}(X, A)$, then $\Delta^M(C_*(x)) \subset \oplus_i C_*(x'_i) \otimes C_*(x''_i)$. Let Δ be the projection of $\Delta^M|_{C_*(x)}$ to the summand group $C_*(x'_k) \otimes C_*(x''_k)$ of $\oplus_i C_*(x'_i) \otimes C_*(x''_i)$. If we can find chain complex isomorphisms that satisfy the following commutative diagram

$$\begin{array}{ccc}
C_*(x) & \xrightarrow{\Delta} & C_*(x'_k) \otimes C_*(x''_k) \\
\parallel & & \parallel \\
\Sigma \tilde{C}_*(K_{\sigma, \omega}) \otimes \mathbb{F}(x) & \xrightarrow{\Delta^R} & \Sigma \tilde{C}_*(K_{\sigma'_k, \omega'_k}) \otimes \Sigma \tilde{C}_*(K_{\sigma''_k, \omega''_k}) \otimes \mathbb{F}(x'_k \otimes x''_k),
\end{array}$$

where Δ^R is as defined in Definition 3.1 and $\mathbb{F}(s)$ is the trivial chain complex generated by s , then the theorem holds.

For $x = a_1 \otimes \cdots \otimes a_m \in \Lambda$ with $x \in T_*^{\sigma, \omega}(X, A)$, suppose $\Delta_k(a_k) = \sum c_k a'_k \otimes a''_k$, where $c_k \in \mathbb{F} \setminus \{0\}$, $a'_k, a''_k \in \mathfrak{k}_k \cup i_k \cup \mathfrak{c}_k$. Then

$$\Delta^T(x) = \sum (-1)^s (c_1 \cdots c_m) (a'_1 \otimes \cdots \otimes a'_m) \otimes (a''_1 \otimes \cdots \otimes a''_m),$$

where $s = \sum_{i>j} |a'_i| |a''_j|$. Suppose $x' = a'_1 \otimes \cdots \otimes a'_m \in T_*^{\sigma', \omega'}(X, A)$ and $x'' = a''_1 \otimes \cdots \otimes a''_m \in T_*^{\sigma'', \omega''}(X, A)$. Denote by $\Delta: C_*(x) \rightarrow C_*(x') \otimes C_*(x'')$ the projection of $\Delta^M|_{C_*(x)}$ to the summand group $C_*(x') \otimes C_*(x'')$ of the direct sum group $\oplus C_*(x') \otimes C_*(x'')$.

Define chain complex $(S_k, d), (T_k, d), (T'_k, d), (T''_k, d)$ for $k = 1, \dots, m$ as follows. S_k is 2-dimensional generated by 1 and ε with $d\varepsilon = 1$ and $|\varepsilon| = 1$, $|1| = 0$. $(T_k, d), (T'_k, d), (T''_k, d)$ are all 1-dimensional chain subcomplex of $(V_*(k), d)$ generated respectively by a_k, a'_k, a''_k . Then $(C_*(x), d)$ is a chain subcomplex of $(S_1 \otimes T_1 \otimes \cdots \otimes S_m \otimes T_m, d \otimes \cdots \otimes d)$ by regarding $b_1 \otimes \cdots \otimes b_m$ as $u_1 \otimes a_1 \otimes \cdots \otimes u_m \otimes a_m$, where $u_k = 1$ if $b_k = a_k$ and $u_k = \varepsilon$ if $db_k = a_k$. Similarly, $(C_*(x'), d), (C_*(x''), d)$ are chain subcomplex of $(S_1 \otimes T'_1 \otimes \cdots \otimes S_m \otimes T'_m, d \otimes \cdots \otimes d), (S_1 \otimes T''_1 \otimes \cdots \otimes S_m \otimes T''_m, d \otimes \cdots \otimes d)$. Define coproduct $\Delta_k^S: S_k \rightarrow S_k \otimes S_k$ and $\Delta_k^T: T_k \rightarrow T'_k \otimes T''_k$ as follows. $\Delta_k^T(a_k) = a'_k \otimes a''_k$. $\Delta_k^S(1) = 1 \otimes 1$, $\Delta_k^S(\varepsilon) = 1 \otimes \varepsilon$ if $a'_k \notin \mathfrak{k}_k$ and $a''_k \in \mathfrak{k}_k$, $\Delta_k^S(\varepsilon) = \varepsilon \otimes 1$ otherwise. Then $\Delta: C_*(x) \rightarrow C_*(x') \otimes C_*(x'')$ is just the restriction of the coproduct homomorphism $\tilde{\Delta}: S_1 \otimes T_1 \otimes \cdots \otimes S_m \otimes T_m \xrightarrow{\Delta_1^S \otimes \Delta_1^T \otimes \cdots \otimes \Delta_m^S \otimes \Delta_m^T} (S_1 \otimes S_1) \otimes (T'_1 \otimes T''_1) \otimes \cdots \otimes (S_m \otimes S_m) \otimes (T'_m \otimes T''_m) \xrightarrow{\Phi} (S_1 \otimes T'_1 \otimes \cdots \otimes S_m \otimes T'_m) \otimes (S_1 \otimes T''_1 \otimes \cdots \otimes S_m \otimes T''_m)$. We have the following commutative diagram

$$\begin{array}{ccc}
S_1 \otimes T_1 \otimes \cdots \otimes S_m \otimes T_m & \xrightarrow{\tilde{\Delta}} & (S_1 \otimes T'_1 \otimes \cdots \otimes S_m \otimes T'_m) \otimes (S_1 \otimes T''_1 \otimes \cdots \otimes S_m \otimes T''_m) \\
\Phi \downarrow & & \Phi \downarrow \\
(S_1 \otimes \cdots \otimes S_m) \otimes (T_1 \otimes \cdots \otimes T_m) & \xrightarrow{\tilde{\Delta}'} & (S_1 \otimes \cdots \otimes S_m) \otimes (S_1 \otimes \cdots \otimes S_m) \otimes (T'_1 \otimes \cdots \otimes T'_m) \otimes (T''_1 \otimes \cdots \otimes T''_m).
\end{array}$$

From the restriction of the above diagram on $C_*(x)$ and $C_*(x') \otimes C_*(x'')$, we have the following diagram

$$\begin{array}{ccc}
C_*(x) & \xrightarrow{\Delta} & C_*(x') \otimes C_*(x'') \\
\Phi \downarrow & & \Phi \downarrow \\
\Sigma \tilde{C}_*(K_{\sigma, \omega}) \otimes \mathbb{F}(x) & \xrightarrow{\Delta^R} & \Sigma \tilde{C}_*(K_{\sigma', \omega'}) \otimes \Sigma \tilde{C}_*(K_{\sigma'', \omega''}) \otimes \mathbb{F}(x' \otimes x''),
\end{array}$$

where Δ^R is as defined in Definition 3.1. It is easy to check that $\Phi: C_*(x) \rightarrow \Sigma \tilde{C}_*(K_{\sigma, \omega}) \otimes \mathbb{F}(x)$ is just the isomorphism $(C_*(x), d) \rightarrow (\Sigma^{|x|+1} \tilde{C}_*(K_{\sigma, \omega}), d_x) \rightarrow (\Sigma^{|x|+1} \tilde{C}_*(K_{\sigma, \omega}), d)$ defined in the proof of Theorem 1.6.

Example 3.5 Let $M = \mathcal{Z}(2^\tau; X, A)$ be as in Example 2.6. We have three coproducts on $H_*(M)$. Δ is the homology coproduct as defined in Theorem 3.4. Δ^T is the coproduct of the generating coalgebra $T_*(2^\tau; X, A) = H_*(Y_1) \otimes \cdots \otimes H_*(Y_m)$. Δ^T depends on the choice of base and is in general not coassociative. $\bar{\Delta} = \Delta^{Y_1} \otimes \cdots \otimes \Delta^{Y_m}$, where Δ^{Y_k} is the homology coproduct of $H_*(Y_k)$. $\bar{\Delta}$ is purely algebraic. Let Δ_k be the coproduct of $H_*(A_k) \cup_{i_k} H_*(X_k)$.

Now we prove $\Delta = \bar{\Delta}$. Let $x = a_1 \otimes \cdots \otimes a_m \in T_*^{\sigma, \omega}(2^\tau; X, A)$. For $k \notin \tau$, $\Delta_k(a_k) = \Delta^{A_k}(a_k) = \Delta^{Y_k}(a_k)$. For $k \in \tau$ and $a_k \in \text{coker } i_k$, $\Delta_k(a_k) = \Delta^{X_k}(a_k) = \Delta^{Y_k}(a_k)$. For $k \in \tau$ and $a_k \in \text{im } i_k$, if $\Delta_k(a_k) \neq \Delta^{X_k}(a_k)$, then $\Delta_k(a_k) - \Delta^{Y_k}(a_k) = \Sigma a'_k \otimes a''_k$, where one or both of a'_k and a''_k is in $\ker i_k$. This implies $(\Delta^T - \bar{\Delta})(x) = \Sigma x' \otimes x''$, $x' \in T_*^{\sigma', \omega'}(X, A)$, $x'' \in T_*^{\sigma'', \omega''}(X, A)$, one or both of $\omega' \cap \tau$ and $\omega'' \cap \tau$ is not empty. By the computation of Example 2.6, $H_*^{\sigma, \omega}(2^\tau) = 0$ if $\omega \cap \tau \neq \phi$ and $H_*^{\sigma, \omega}(2^\tau) = \mathbb{F}$ if $\omega \cap \tau = \phi$. So the restricted coproduct $\Delta^R: H_*^{\sigma, \omega}(K) \rightarrow H_*^{\sigma', \omega'}(K) \otimes H_*^{\sigma'', \omega''}(K)$ is 0 if $\omega' \cap \tau \neq \phi$ or $\omega'' \cap \tau \neq \phi$ and $\Delta^R: H_*^{\sigma, \omega}(K) \rightarrow H_*^{\sigma', \omega'}(K) \otimes H_*^{\sigma'', \omega''}(K)$ is an isomorphism if $\omega' \cap \tau = \phi$ and $\omega'' \cap \tau = \phi$ ($\Delta^R(\phi) = \phi \otimes \phi$). By Theorem 3.4, $\Delta(x) = \bar{\Delta}(x)$.

4 Diagonal tensor product of (co)algebras

Definition 4.1 Let everything be as in Definition 1.3 and Definition 3.1.

The universal coproduct $\Delta^K: H_*(K; \Sigma, \Omega) \rightarrow H_*(K; \Sigma, \Omega) \otimes H_*(K; \Sigma, \Omega)$ is defined as follows. For $(\sigma, \omega) \in I(K; \Sigma, \Omega)$ and $x \in H_*^{\sigma, \omega}(K)$, $\Delta^K(x) = \Sigma \Delta_{\sigma, \omega}^{\sigma', \omega', \sigma'', \omega''}(x)$, where the sum is taken over all $\sigma' \cup \sigma'' \subset \sigma$, $\omega \subset \omega' \cup \omega''$ (empty set is allowed), $(\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$ and $\Delta_{\sigma, \omega}^{\sigma', \omega', \sigma'', \omega''}: H_*^{\sigma, \omega}(K) \rightarrow H_*^{\sigma', \omega'}(K) \otimes H_*^{\sigma'', \omega''}(K)$ is the restricted coproduct Δ^R in Definition 3.1. $(H_*(K; \Sigma, \Omega), \Delta^K)$ is called the universal system coalgebra of K with index set $I(K; \Sigma, \Omega)$.

Dually, the universal product $\Pi_K: H^*(K; \Sigma, \Omega) \otimes H^*(K; \Sigma, \Omega) \rightarrow H^*(K; \Sigma, \Omega)$ is defined as follows. For $(\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$ and $x' \in H_{\sigma', \omega'}^*(K)$, $x'' \in H_{\sigma'', \omega''}^*(K)$, $\Pi_K(x', x'') = \Sigma \Pi_{\sigma', \omega', \sigma'', \omega''}^{\sigma, \omega}(x', x'')$, where the sum is taken over all $\sigma' \cup \sigma'' \subset \sigma$, $\omega \subset \omega' \cup \omega''$, $(\sigma, \omega) \in I(K; \Sigma, \Omega)$ and $\Pi_{\sigma', \omega', \sigma'', \omega''}^{\sigma, \omega}: H_{\sigma', \omega'}^*(K) \otimes H_{\sigma'', \omega''}^*(K) \rightarrow H_{\sigma, \omega}^*(K)$ is the restricted product Π_R in Definition 3.1. $(H^*(K; \Sigma, \Omega), \Pi_K)$ is called the universal system algebra of K with index set $I(K; \Sigma, \Omega)$.

The normal coproduct $\tilde{\Delta}^K: H_*(K; \Sigma, \Omega) \rightarrow H_*(K; \Sigma, \Omega) \otimes H_*(K; \Sigma, \Omega)$ is defined as follows. For $(\sigma, \omega) \in I(K; \Sigma, \Omega)$

and $x \in H_*^{\sigma, \omega}(K)$, $\tilde{\Delta}^K(x) = \Sigma \Delta_{\sigma, \omega}^{\sigma', \omega', \sigma'', \omega''}(x)$, where the sum is taken over all $\sigma' \cup \sigma'' \subset \sigma$, $\omega = \omega' \cup \omega''$, $(\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$. $(H_*(K; \Sigma, \Omega), \tilde{\Delta}^K)$ is called the normal system coalgebra of K with index set $I(K; \Sigma, \Omega)$. $(H_*(K; \phi, [m]), \tilde{\Delta}^K)$ is a coassociative, cocommutative graded algebra with unit (see the remark after Theorem 4.7).

Dually, the normal product $\tilde{\Pi}_K: H^*(K; \Sigma, \Omega) \otimes H^*(K; \Sigma, \Omega) \rightarrow H^*(K; \Sigma, \Omega)$ is defined as follows. For $(\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$ and $x' \in H_{\sigma', \omega'}^*(K)$, $x'' \in H_{\sigma'', \omega''}^*(K)$, $\tilde{\Pi}_K(x', x'') = \Pi_{\sigma', \omega', \sigma'', \omega''}^{\sigma, \omega}(x', x'')$, where the sum is taken over all $\sigma' \cup \sigma'' \subset \sigma$, $\omega = \omega' \cup \omega''$ and $(\sigma, \omega) \in I(K; \Sigma, \Omega)$. $(H^*(K; \Sigma, \Omega), \tilde{\Pi}_K)$ is called the normal system algebra of K with index set $I(K; \Sigma, \Omega)$. $(H^*(K; \phi, [m]), \tilde{\Pi}_K)$ is an associative, commutative graded algebra with unit.

The special coproduct $\overline{\Delta}^K: H_*(K; \Sigma, \Omega) \rightarrow H_*(K; \Sigma, \Omega) \otimes H_*(K; \Sigma, \Omega)$ is defined as follows. For $(\sigma, \omega) \in I(K; \Sigma, \Omega)$ and $x \in H_*^{\sigma, \omega}(K)$, $\overline{\Delta}^K(x) = \Sigma \Delta_{\sigma, \omega}^{\sigma', \omega', \sigma'', \omega''}(x)$, where the sum is taken over all $\sigma = \sigma' \sqcup \sigma''$, $\omega = \omega' \sqcup \omega''$ (\sqcup disjoint union). $(H_*(K; \Sigma, \Omega), \overline{\Delta}^K)$ is called the special system coalgebra of K with index set $I(K; \Sigma, \Omega)$. It is a coassociative, cocommutative graded algebra with unit (see Example 4.6).

Dually, the special product $\overline{\Pi}_K: H^*(K; \Sigma, \Omega) \otimes H^*(K; \Sigma, \Omega) \rightarrow H^*(K; \Sigma, \Omega)$ is defined as follows. For $(\sigma', \omega'), (\sigma'', \omega'') \in I(K; \Sigma, \Omega)$ and $x' \in H_{\sigma', \omega'}^*(K)$, $x'' \in H_{\sigma'', \omega''}^*(K)$, $\overline{\Pi}_K(x', x'') = 0$ if $\sigma' \cap \sigma'' \neq \phi$, or $\omega' \cap \omega'' \neq \phi$, or $(\sigma' \cup \sigma'', \omega' \cup \omega'') \notin I(K; \Sigma, \Omega)$ and $\overline{\Pi}_K(x', x'') = \Pi_{\sigma', \omega', \sigma'', \omega''}^{\sigma, \omega}(x', x'')$, if $\sigma' \cap \sigma'' = \phi$, $\omega' \cap \omega'' = \phi$ and $(\sigma, \omega) \in I(K; \Sigma, \Omega)$, where $\sigma = \sigma' \cup \sigma''$, $\omega = \omega' \cup \omega''$. $(H^*(K; \Sigma, \Omega), \overline{\Pi}_K)$ is called the special system algebra of K with index set $I(K; \Sigma, \Omega)$. It is an associative, commutative graded algebra with unit.

Definition 4.2 Let $(A, \Delta^A), (B, \Delta^B)$ be two coalgebras such that both A and B are systems with the same index set Λ , then their diagonal tensor product coalgebra $(A, \Delta^A) \hat{\otimes} (B, \Delta^B) = (A \hat{\otimes} B, \Delta^{A \hat{\otimes} B})$ is defined as follows. The diagonal tensor product group $A \hat{\otimes} B$ is a subgroup of $A \otimes B$. Denote the inclusion by i . $A \hat{\otimes} B$ is also a quotient group of $A \otimes B$ over the subgroup $A \nabla B = \oplus_{\alpha, \beta \in \Lambda, \alpha \neq \beta} A_\alpha \otimes B_\beta$. Denote the quotient homomorphism by j . Then $\Delta^{A \hat{\otimes} B} = (j \otimes j)(\Delta^{A \otimes B})i$, as shown in the following diagram

$$\Delta^{A \hat{\otimes} B}: A \hat{\otimes} B \xrightarrow{i} A \otimes B \xrightarrow{\Delta^A \otimes \Delta^B} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\Phi} (A \otimes B) \otimes (A \otimes B) \xrightarrow{j \otimes j} (A \hat{\otimes} B) \otimes (A \hat{\otimes} B).$$

Dually, let $(A, \Pi_A), (B, \Pi_B)$ be two algebras such that both A and B are systems with the same index set Λ , then their diagonal tensor product algebra $(A, \Pi_A) \hat{\otimes} (B, \Pi_B) = (A \hat{\otimes} B, \Pi_{A \hat{\otimes} B})$ is defined as follows. $\Pi_{A \hat{\otimes} B} = j(\Pi_{A \otimes B})(i \otimes i)$, as shown in the following diagram

$$\Pi_{A \hat{\otimes} B}: (A \hat{\otimes} B) \otimes (A \hat{\otimes} B) \xrightarrow{i \otimes i} (A \otimes B) \otimes (A \otimes B) \xrightarrow{\Phi} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\Pi_A \otimes \Pi_B} A \otimes B \xrightarrow{j} (A \hat{\otimes} B).$$

With above definitions, Theorem 3.4 can be restated as follows.

Theorem 4.3 Let everything be as in Theorem 1.6.

The homology coalgebra of M is isomorphic to the diagonal tensor product of the universal system coalgebra of K with index set $I(K; X, A)$ and the homology generating coalgebra of K with respect to (X, A) . Dually, the cohomology algebra of M is isomorphic to the diagonal tensor product of the universal system algebra of K with index set $I(K; X, A)$

and the cohomology generating algebra of K with respect to (X, A) . Precisely,

$$(H_*(M), \Delta) \cong (H_*(K; \Sigma, \Omega), \Delta^K) \widehat{\otimes} (T_*(K; X, A), \Delta^T), \quad (H^*(M), \Pi) \cong (H^*(K; \Sigma, \Omega), \Pi_K) \widehat{\otimes} (T^*(K; X, A), \Pi_T).$$

If K is a simplicial subcomplex of L and $i, i_{\sigma, \omega}$ are as in Theorem 1.6, then the homomorphism

$$i_S: H_*(K; \Sigma, \Omega) = \bigoplus_{(\sigma, \omega) \in I(K; X, A)} H_{\sigma, \omega}^{\sigma, \omega}(K) \xrightarrow{\bigoplus i_{\sigma, \omega}} \bigoplus_{(\sigma, \omega) \in I(L; X, A)} H_{\sigma, \omega}^{\sigma, \omega}(L) = H_*(L; \Sigma, \Omega)$$

is a coalgebra homomorphism and the inclusion $i_T: T_*(K; X, A) \rightarrow T_*(L; X, A)$ is a monomorphism of coalgebras.

$i = i_S \widehat{\otimes} i_T = \bigoplus_{(\sigma, \omega) \in I(K; X, A)} i_{\sigma, \omega} \otimes 1$. Dually,

$$i_S^*: H^*(L; \Sigma, \Omega) = \bigoplus_{(\sigma, \omega) \in I(L; X, A)} H_{\sigma, \omega}^{\sigma, \omega}(L) \xrightarrow{\bigoplus i_{\sigma, \omega}^*} \bigoplus_{(\sigma, \omega) \in I(K; X, A)} H_{\sigma, \omega}^{\sigma, \omega}(K) = H^*(K; \Sigma, \Omega)$$

is an algebra homomorphism and $i_T^*: T^*(L; X, A) \rightarrow T^*(K; X, A)$ is an epimorphism of algebras. $i^* = i_S^* \widehat{\otimes} i_T^* =$

$$\bigoplus_{(\sigma, \omega) \in I(K; X, A)} i_{\sigma, \omega}^* \otimes 1.$$

Proof Since the restricted coproduct is natural, i.e., the following diagram

$$\begin{array}{ccc} H_{\sigma, \omega}^{\sigma, \omega}(K) & \xrightarrow{\Delta^R} & H_{\sigma', \omega'}^{\sigma', \omega'}(K) \otimes H_{\sigma'', \omega''}^{\sigma'', \omega''}(K) \\ i_{\sigma, \omega} \downarrow & & i_{\sigma', \omega'} \otimes i_{\sigma'', \omega''} \downarrow \\ H_{\sigma, \omega}^{\sigma, \omega}(L) & \xrightarrow{\Delta^R} & H_{\sigma', \omega'}^{\sigma', \omega'}(L) \otimes H_{\sigma'', \omega''}^{\sigma'', \omega''}(L) \end{array}$$

is commutative, i_S is a coalgebra homomorphism.

Theorem 4.4 *Let everything be as in Theorem 1.6. If for $k = 1, \dots, m$, there is a subcoalgebra \mathbf{i}_k of $H_*(A_k)$ such that $H_*(A_k) = \mathbf{i}_k \oplus \ker i_k$. Then the homology generating coalgebra $(T_*(K; X, A), \Delta^T)$ is a coassociative, cocommutative graded coalgebra with unit and the cohomology generating algebra $(T^*(K; X, A), \Pi_T)$ is an associative, commutative graded algebra with unit. The universal (co)algebra in the diagonal tensor product can be replaced by the normal (co)algebra, i.e.,*

$$(H_*(M), \Delta) \cong (H_*(K; \Sigma, \Omega), \tilde{\Delta}^K) \widehat{\otimes} (T_*(K; X, A), \Delta^T), \quad (H^*(M), \Pi) \cong (H^*(K; \Sigma, \Omega), \tilde{\Pi}_K) \widehat{\otimes} (T^*(K; X, A), \Pi_T).$$

Proof The coproduct Δ_k of $H_*(A_k) \cup_{i_k} H_*(X_k)$ satisfies $\Delta_k|_{H_*(A_k)} = \Delta^{H_*(A_k)}$ and $\Delta_k|_{H_*(X_k)} = \Delta^{H_*(X_k)}$. So $H_*(A_k) \cup_{i_k} H_*(X_k)$ is a coassociative and cocommutative graded coalgebra with unit. Since $\text{coim } i_k$ is a subcoalgebra of $H_*(A_k)$, Δ^T satisfies that if $x \in T_{\sigma, \omega}^{\sigma, \omega}(X, A)$ and $\Delta^T(x) = \sum_i x'_i \otimes x''_i$ with $x'_i \in T_{\sigma'_i, \omega'_i}^{\sigma'_i, \omega'_i}(X, A)$ and $x''_i \in T_{\sigma''_i, \omega''_i}^{\sigma''_i, \omega''_i}(X, A)$, then $\sigma'_i \cup \sigma''_i \subset \sigma$ and $\omega = \omega'_i \cup \omega''_i$. So the universal coproduct can be replaced by normal coproduct.

Theorem 4.5 *Let everything be as in Theorem 1.6. If for $k = 1, \dots, m$, $H_*(A_k)$ and $H_*(X_k)$ are primitive coalgebras, i.e., $\Delta(\iota) = \iota \otimes \iota$ for the unit ι of both coalgebras and $\Delta(x) = \iota \otimes x + x \otimes \iota$ for all other base element x . Then by Theorem 4.4, $(T_*(K; X, A), \Delta^T)$ is coassociative, cocommutative, with unit and $(T^*(K; X, A), \Pi_T)$ is associative, commutative, with unit. The normal (co)algebra in the diagonal tensor product can be replaced by the special (co)algebra, i.e.,*

$$(H_*(M), \Delta) \cong (H_*(K; \Sigma, \Omega), \overline{\Delta}^K) \widehat{\otimes} (T_*(K; X, A), \Delta^T), \quad (H^*(M), \Pi) \cong (H^*(K; \Sigma, \Omega), \overline{\Pi}_K) \widehat{\otimes} (T^*(K; X, A), \Pi_T).$$

Proof $H_*(A_k) \cup_{i_k} H_*(X_k)$ is also a primitive coalgebra. So Δ^T satisfies that if $x \in T_{\sigma, \omega}^{\sigma, \omega}(X, A)$ and $\Delta^T(x) = \sum_i x'_i \otimes x''_i$ with $x'_i \in T_{\sigma'_i, \omega'_i}^{\sigma'_i, \omega'_i}(X, A)$ and $x''_i \in T_{\sigma''_i, \omega''_i}^{\sigma''_i, \omega''_i}(X, A)$, then $\sigma = \sigma'_i \sqcup \sigma''_i$ and $\omega = \omega'_i \sqcup \omega''_i$.

Example 4.6 The most general polyhedral product satisfying the condition of Theorem 4.5 is the polyhedral product $\mathcal{Z}(K; SX, SA)$, where $S = S^1 \wedge$ is the suspension of a space with base point and $\mathcal{Z}(K; X, A)$ is a polyhedral product satisfying the condition of Theorem 1.6. As ungraded groups, $H_*(\mathcal{Z}(K; SX, SA))$ and $H_*(\mathcal{Z}(K; X, A))$ are isomorphic, but the coproduct may be very different.

The cohomology generating algebra $(T_*(K; SX, SA), \Pi_T)$ is a very interesting algebra. We will study a typical case.

Let K be a simplicial complex with vertex set a subset of $[m]$ and Σ, Ω be two subsets of $[m]$. Define polyhedral product $M = \mathcal{Z}(K; X, A)$, $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ as follows. $(X_k, A_k) = (S^2, *)$ if $k \notin \Sigma$, where $*$ is the base point of S^2 ; $(X_k, A_k) = (D^3, S^2)$ if $k \in \Sigma$ but $k \notin \Omega$, where S^2 is the boundary of D^3 ; $(X_k, A_k) = (S^4, S^2)$ if $k \in \Sigma \cap \Omega$, where $\theta: S^2 \rightarrow S^4$ is the inclusion map defined in Example 1.8. Since $T_*(X, A)$ is evenly graded, we regard it as ungraded. All $\ker i_k$, $\text{coker } i_k$, $\text{im } i_k = \text{coim } i_k$ are one dimensional and suppose a fixed generator is given. For base element $x = a_1 \otimes \cdots \otimes a_m \in T_*(X, A)$, let $\sigma = \{k | a_k \in \text{coker } i_k\}$ and $\omega = \{k | a_k \in \ker i_k\}$. Then the correspondence $x \rightarrow (\sigma, \omega)$ is a 1-1 correspondence from the base of $T_*(K; X, A)$ to $I(K; \Sigma, \Omega)$. We identify the two objects by the correspondence, i.e., $I(K; \Sigma, \Omega)$ is a base of $T_*(K; X, A)$ with coproduct Δ^T defined as follows. For $(\sigma, \omega) \in I(K; \Sigma, \Omega)$, $\Delta^T(\sigma, \omega) = \Sigma(\sigma', \omega') \otimes (\sigma'', \omega'')$, where the sum is taken over all $\sigma' \sqcup \sigma'' = \sigma$, $\omega' \sqcup \omega'' = \omega$. Dually, $T^*(K; X, A)$ is an algebra with base $I(K; \Sigma, \Omega)$ and product defined by $(\sigma', \omega')(\sigma'', \omega'') = 0$ if $\sigma' \cap \sigma'' \neq \emptyset$, or $\omega' \cap \omega'' \neq \emptyset$, or $(\sigma' \cup \sigma'', \omega' \cup \omega'') \notin I(K; \Sigma, \Omega)$ and $(\sigma', \omega')(\sigma'', \omega'') = (\sigma' \cup \sigma'', \omega' \cup \omega'')$ otherwise.

By Theorem 4.5,

$$\begin{aligned} (H_*(M), \Pi) &\cong (H_*(K; \Sigma, \Omega), \overline{\Delta}^K) \widehat{\otimes} (T_*(K; X, A), \Delta^T) \cong (H_*(K; \Sigma, \Omega), \overline{\Delta}^K), \\ (H^*(M), \Pi) &\cong (H^*(K; \Sigma, \Omega), \overline{\Pi}_K) \widehat{\otimes} (T^*(K; X, A), \Pi_T) \cong (H^*(K; \Sigma, \Omega), \overline{\Pi}_K). \end{aligned}$$

This shows that the special system algebra (coalgebra) is isomorphic to the cohomology algebra (homology coalgebra) of a polyhedral product and so is an (co)associative, (co)commutative graded algebra with unit.

Theorem 4.7 Let K be a simplicial complex with vertex set a subset of $[m]$ and $M = \mathcal{Z}(K; D^1, S^0)$ be the polyhedral product such that $(X_k, A_k) = (D^1, S^0)$ for $k = 1, \dots, m$. Then the homology coalgebra (cohomology algebra) of M is isomorphic to the diagonal tensor product of the normal system coalgebra (algebra) and the generating system coalgebra (algebra). Precisely,

$$(H_*(M), \Delta) \cong (H_*(K; \phi, [m]), \tilde{\Delta}^K) \widehat{\otimes} (\tilde{A}_*[m], \Delta), \quad (H^*(M), \Pi) \cong (H^*(K; \phi, [m]), \tilde{\Pi}_K) \widehat{\otimes} (\tilde{A}^*[m], \Pi),$$

where $(\tilde{A}_*[m], \Delta)$ and $(\tilde{A}^*[m], \Pi)$ are dual to each other and the algebra $\tilde{A}^*[m]$ is defined as follows. The base of $\tilde{A}^*[m]$ is $2^{[m]}$ with degree 0 for all. For subsets ω' and ω'' of $[m]$, $\Pi(\omega', \omega'') = \omega' \omega'' = (-1)^{|\omega' \cap \omega''|} \omega' \cup \omega''$, where $|\omega' \cap \omega''|$ is the cardinality of $\omega' \cap \omega''$.

For moment-angle complex $M = \mathcal{Z}(K; D^2, S^1)$ such that $(X_k, A_k) = (D^2, S^1)$ for $k = 1, \dots, m$, the homology coalgebra (cohomology algebra) of M is isomorphic to the diagonal tensor product of the special system coalgebra (algebra) and the generating system coalgebra (algebra). Precisely,

$$(H_*(M), \Delta) \cong (H_*(K; \phi, [m]), \overline{\Delta}^K) \widehat{\otimes} (\overline{A}_*[m], \Delta), (H^*(M), \Pi) \cong (H^*(K; \phi, [m]), \overline{\Pi}_K) \widehat{\otimes} (\overline{A}^*[m], \Pi),$$

where $(\overline{A}_*[m], \Delta)$ and $(\overline{A}^*[m], \Pi)$ are dual to each other and the algebra $\overline{A}^*[m]$ is defined as follows. The base of $\overline{A}^*[m]$ is $2^{[m]}$ with degree the cardinality of the subset. For subsets ω' and ω'' of $[m]$, $\Pi(\omega', \omega'') = 0$ if $\omega' \cap \omega'' \neq \phi$ and $\Pi(\omega', \omega'') = \omega' \omega'' = (-1)^\tau \omega' \cup \omega''$, where $(-1)^\tau$ is the sign of the permutation from $\omega' \cup \omega''$ (ω', ω'' ordered) to its ordered set. Precisely, if $\omega' = \{j_1, \dots, j_u\}$, $\omega'' = \{k_{u+1}, \dots, k_s\}$, $\omega' \cup \omega'' = \{i_1, \dots, i_s\}$, $i_1 < \dots < i_s$, $j_1 < \dots < j_u$, $k_{u+1} < \dots < k_s$, then $(-1)^\tau$ is the sign of the permutation $\begin{pmatrix} i_1 & \dots & i_u & i_{u+1} & \dots & i_s \\ j_1 & \dots & j_u & k_{u+1} & \dots & k_s \end{pmatrix}$.

Proof $H_*(S^0)$ is 2-dimensional generated by its two isolated points x_1, x_2 . Take the unit to be $x_1 \in \text{coim } i_k = \text{im } i_k$, then $x_1 - x_2$ is a generator of $\ker i_k$. Since $\Delta(x_1 - x_2) = x_1 \otimes x_1 - x_2 \otimes x_2 = (x_1 - x_2) \otimes x_1 + (x_2 - x_1) \otimes (x_1 - x_2) + x_1 \otimes (x_1 - x_2)$, all $H_*(A_k) \cup_{i_k} H_*(X_k)$ is the coalgebra (\tilde{B}_*, Δ) defined as follows. \tilde{B}_* is 2-dimensional generated by unit ι and κ with $\Delta(\iota) = \iota \otimes \iota$, $\Delta(\kappa) = \iota \otimes \kappa + \kappa \otimes \iota - \kappa \otimes \kappa$ and $|\iota| = |\kappa| = 0$. M satisfies the condition of Theorem 4.4. Dually, \tilde{B}^* is 2-dimensional generated by unit 1 and ε and the product $(\Pi(a, b)$ is denoted by ab) satisfies $\varepsilon^2 = -\varepsilon$. So the cohomology generating algebra $T^*(K; X, A)$ is isomorphic to $\tilde{B}^* \otimes \dots \otimes \tilde{B}^*$ (m -fold) which is isomorphic to $\tilde{A}^*[m]$ by corresponding $\{i_1, \dots, i_s\}$ in $\tilde{A}^*[m]$ to $u_1 \otimes \dots \otimes u_m$ in $\tilde{B}^* \otimes \dots \otimes \tilde{B}^*$, where $u_{i_k} = \varepsilon$ for $k = 1, \dots, s$ and $u_j = 1$ otherwise.

Similarly, $H_*(S^1)$ is 2-dimensional generated by unit $|\iota| = 0$ and $|\kappa| = 1$. So all $H_*(A_k) \cup_{i_k} H_*(X_k)$ is the coalgebra (\overline{B}_*, Δ) defined as follows. \overline{B}_* is 2-dimensional generated by unit ι and $\kappa \in \ker i_k$ such that $\Delta(\iota) = \iota \otimes \iota$, $\Delta(\kappa) = \iota \otimes \kappa + \kappa \otimes \iota$. M satisfies the condition of Theorem 4.5. Dually, \overline{B}^* is 2-dimensional generated by unit 1 and ε and the product satisfies $\varepsilon^2 = 0$. So the cohomology generating algebra $T^*(K; X, A)$ is isomorphic to $\overline{B}^* \otimes \dots \otimes \overline{B}^*$ (m -fold) which is isomorphic to $\overline{A}^*[m]$ by corresponding $\{i_1, \dots, i_s\}$ in $\overline{A}^*[m]$ to $u_1 \otimes \dots \otimes u_m$ in $\overline{B}^* \otimes \dots \otimes \overline{B}^*$, where $u_{i_k} = \varepsilon$ for $k = 1, \dots, s$ and $u_j = 1$ otherwise.

Remark Notice that the coproducts of $(H_*(K; \phi, [m]), \tilde{\Delta}^K) \widehat{\otimes} (\tilde{A}_*([m]), \Delta)$ and $(H_*(K; \phi, [m]), \tilde{\Delta}^K)$ differ only in sign. The coassociativity and cocommutativity of $(H_*(K; \phi, [m]), \tilde{\Delta}^K) \widehat{\otimes} (\tilde{A}_*([m]), \Delta)$ implies that of $(H_*(K; \phi, [m]), \tilde{\Delta}^K)$. So the normal system coalgebra $(H_*(K; \phi, [m]), \tilde{\Delta}^K)$ is a coassociative, cocommutative graded coalgebra with unit and the normal system algebra $(H^*(K; \phi, [m]), \tilde{\Pi}_K)$ is an associative, commutative graded algebra with unit.

There is an algebra isomorphism from the special system algebra $(H^*(K; \phi, [m]), \overline{\Pi}_K)$ to $\text{Tor}_*^{\mathbb{F}[\mathbf{x}]}(\mathbb{F}, F(K))$, where $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \dots, x_m]$ is the polynomial algebra generated by x_1, \dots, x_m and $F(K)$ is the Stanley-Reisner face ring of K (see [5]). But the degree of the isomorphism is not natural. The following example shows that the normal system algebra $(H^*(K; \phi, [m]), \tilde{\Pi}_K)$ is not isomorphic to $\text{Tor}_*^{\mathbb{F}[\mathbf{x}]}(\mathbb{F}, F(K))$ even as an ungraded algebra.

Example 4.8 Let K be the pentagon, i.e., the vertex set of K is $[5]$ with edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}$. Then $H_*(K; \phi, [5]) = \oplus_{\omega \subset [5]} H_*^{\phi, \omega}(K)$. Let $\omega' = \{1, 3, 5\}$ and $\omega'' = \{2, 4, 5\}$. $H_*^{\phi, \omega'}(K)$ is one dimensional at degree 1 generated by $\{1\} - \{3\}$; $H_*^{\phi, \omega''}(K)$ is one dimensional at degree 1 generated by $\{2\} - \{4\}$; $H_*^{\phi, [5]}(K) = H_*(K)$ is one

dimensional at degree 2 generated by $\Sigma_{i=1}^4 \{i, i+1\} - \{1, 5\}$. By definition, the restricted coproduct $\Delta^R: H_*^{\phi, [5]}(K) \rightarrow H_*^{\phi, \omega'}(K) \otimes H_*^{\phi, \omega''}(K)$ is

$$\begin{aligned}
& \Delta^R(\{1, 2\} + \{2, 3\} + \{3, 4\} + \{4, 5\} - \{1, 5\}) \\
&= \{1\} \otimes \{2\} - \{3\} \otimes \{2\} + \{3\} \otimes \{4\} - \{5\} \otimes \{4\} - \{1, 5\} \otimes \phi \\
&= (\{1\} - \{3\}) \otimes \{2\} + (\{3\} - \{1\}) \otimes \{4\} + (\{1\} - \{5\}) \otimes \{4\} - \{1, 5\} \otimes \phi \\
&\sim (\{1\} - \{3\}) \otimes (\{2\} - \{4\})
\end{aligned}$$

This shows that the restricted product $\Pi_R: H_{\phi, \omega'}^*(K) \otimes H_{\phi, \omega''}^*(K) \rightarrow H_{\phi, [5]}^*(K)$ is non-trivial.

Geometrically, let $M = (K; D^1, S^0)$ be as in Theorem 4.7 with K a pentagon. Denote by α, β, γ the generator dual to $\{1\} - \{3\}, \{2\} - \{4\}, \Sigma_{i=1}^4 \{i, i+1\} - \{1, 5\}$ defined above. By Theorem 4.7, $\alpha \otimes \{1, 3, 5\}, \beta \otimes \{2, 4, 5\}, \gamma \otimes [5]$ are all non-trivial cohomology classes of $H^*(M)$. The above computation shows that the cup product of $H^*(M)$ satisfies that

$$(\alpha \otimes \{1, 3, 5\}) \cup (\beta \otimes \{2, 4, 5\}) = -\gamma \otimes [5],$$

where the sign comes from $(-1)^{|\omega' \cap \omega''|}$.

For moment-angle complex $M = (K; D^2, S^1)$ with K a pentagon, we still have non-trivial cohomology classes $\alpha \otimes \{1, 3, 5\}, \beta \otimes \{2, 4, 5\}, \gamma \otimes [5]$. By Theorem 4.7, $(\alpha \otimes \{1, 3, 5\}) \cup (\beta \otimes \{2, 4, 5\}) = 0$ since $\{1, 3, 5\} \cap \{2, 4, 5\} \neq \phi$.

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